Riemann Hypothesis: Backward Parabolic Positivity Barriers for the Xi Flow

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Abstract

We develop a parabolic positivity method for the Riemann Ξ-function based on backward Carleman–Kato barriers for the de Bruijn heat flow (*the Xi flow*). Writing $E_t = e^{-t\partial_z^2}\Xi$ for the de Bruijn deformation of the completed zeta function, we construct a backward barrier that propagates

$$\Im\big(- E_t'(z)/E_t(z)\big) \ \geq 0 \qquad (z \in \mathbb{C}_+)$$

from de Bruijn's real-zero time t_+ back to $t = 0$.

The barrier is built using a time-local tube around the moving zeros, with a weight that vanishes algebraically of order $\gamma > 2$, suppressing the poles of $-E_t'/E_t$ without dividing by a vanishing factor. Inside this tube, we introduce a relative-derivative calculus that forces all commutator errors to carry a factor w^2 and be dominated by the positive Carleman bulk $\phi' w^2$. Thin time shoulders allow the backward Carleman estimate to act on an interior slab, and a collision-bridging lemma, based on dominated convergence, propagates positivity across all collision times.

On each collision-free time component, we close the Carleman absorption under an explicit cubic relation between the Carleman parameter and the tube thickness, with constants depending only on an upper bound for the zero speeds. Iterating from de Bruijn's real-zero time t_+ yields $\Im(-\Xi'/\Xi) \geq 0$ on $\mathbb{C}_+$ at $t = 0$. A Pick-positivity argument then shows that $-\Xi'/\Xi$ has no poles in $\mathbb{C}_+$, so Ξ has no zeros in $\mathbb{C}_+$. By evenness, all nontrivial zeros lie on the critical line. Combined with the Rodgers–Tao lower bound $\Lambda \geq 0$ for the de Bruijn–Newman constant, this gives $\Lambda = 0$, and hence the Riemann Hypothesis.

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1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re s = \frac{1}{2}$. Writing $\Xi(z) = \xi(\frac{1}{2} + iz)$ for the completed zeta function, RH is equivalent to:

all zeros of the entire function Ξ lie on the real axis.

De Bruijn introduced the heat flow

$$E_t(z) := (e^{-t\partial_z^2}\Xi)(z), \qquad t \in \mathbb{R},$$

which we will refer to as the *Xi flow* (i.e., the de Bruijn heat deformation of the Riemann Ξ-function; not the holomorphic vector-field ODE $\dot{z} = \xi(z)$). He showed that there is a real parameter Λ (the de Bruijn–Newman constant) with the property that E_t has only real zeros for all $t \geq \Lambda$, and moreover that once all zeros of E_{t_0} are real at some time t_0, they remain real for all later times $t \geq t_0$ (see [2, 3, 6, 7]). Rodgers and Tao proved the lower bound $\Lambda \geq 0$ [8]. The conjectural equality $\Lambda = 0$ is equivalent to RH and can be read as: *the first time at which the heat flow E_t acquires only real zeros is already $t = 0$.*

This paper develops a *parabolic positivity* method that approaches RH through the de Bruijn heat flow (the *Xi flow*). We work with the logarithmic derivative

$$g_t(z) \; := \; -\frac{E_t'(z)}{E_t(z)} \; = \; p_t(x, y) + i\, h_t(x, y), \qquad z = x + iy,$$

and its imaginary part h_t. On pole-free regions g_t solves a complex Burgers-type equation and h_t satisfies a drift–diffusion PDE in the x–direction. Our main analytic input is a backward Carleman–Kato barrier. At the level of the local estimate (Proposition 6.3), this is a *conditional* barrier: it propagates the *Pick positivity*

$$\Im\big(-E_t'(z)/E_t(z)\big) \; \geq 0 \qquad (z \in \mathbb{C}_+)$$

from a top time t_1 *backwards* on a short collision-free window, under the hypothesis that $\Im(-E_{t_1}'/E_{t_1}) \geq 0$ there. Starting from a real-zero time t_+ supplied by de Bruijn, and combining this conditional barrier with a collision-bridging argument and an exhaustion in (x, y), we propagate $\Im(-E_t'/E_t) \geq 0$ all the way down to $t = 0$. A short Pick-positivity argument then shows that $-\Xi'/\Xi$ has no poles in the upper half-plane, hence Ξ has only real zeros. In view of the definition of Λ and de Bruijn's monotonicity theorem recalled above, this forces the de Bruijn–Newman constant to satisfy $\Lambda \leq 0$. Combined with the Rodgers–Tao lower bound $\Lambda \geq 0$ [8], this yields $\Lambda = 0$ and RH.

The barrier relies on three ingredients:

- a *time–local, algebraically vanishing tube cutoff* $\Theta_{\rho,\varepsilon,\gamma}(t, z)$ concentrating near the instantaneous zero set of E_t, with $\gamma > 2$ and $\varepsilon = \theta\rho$;
- a backward Carleman estimate in x with a time-blowing-up weight, coupled to a Kato inequality for $h^- = \max\{-h, 0\}$;
- *relative-derivative* control of the weight on the tube: on the support we only ever see $\frac{|\partial_x W|}{W}$, $\frac{|\partial_x^2 W|}{W}$, $\frac{|\partial_t W|}{W}$, so every commutator error carries the same $w^2 = (Wh^-)^2$ factor as the positive Carleman term.

The tube cutoff is *time-local* (it follows the moving zeros), *algebraically vanishing* (it kills the logarithmic singularities of g_t near zeros without ever making the weight literally zero), and its derivatives are controlled on collision-free windows by the local speed of the zero set. The absorption closes under a cubic coupling between the Carleman parameter, the spatial tube radius, and the time length of the window. A collision-bridging argument, based on a tube-weighted energy, lets us iterate across the finitely many collision times on a compact interval.

1.1 External input: de Bruijn real-zero slice

A theorem of de Bruijn (see [2, 3]; see also Newman [6]) implies the existence of a time with only real zeros:

$$\exists\, t_+ > 0 \quad \text{such that} \quad E_{t_+}(z) \neq 0 \text{ for all } z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{1.1}$$

We do not use any finer structure at t_+ (e.g. simplicity or spacing). In Section 4 we prove a general lemma (Lemma 4.8) that any even real entire Cartwright function with only real zeros satisfies $\Im(-F'/F) > 0$ on $\mathbb{C}_+$. Applying this to $F = E_{t_+}$ yields

$$\Im\big(- E'_{t_+}(z)/E_{t_+}(z)\big) \;>\; 0 \qquad (z \in \mathbb{C}_+),$$

which we take as initial data for the backward barrier.

2 Main Theorem

Contribution. We study the *Xi flow*, namely the de Bruijn heat deformation $E_t = e^{-t\partial_z^2}\Xi$, together with a backward Carleman–Kato barrier built on time-local algebraic tubes. This barrier propagates the Pick-positivity condition $\Im(-E_t'/E_t) \geq 0$ on $\mathbb{C}_+$ from de Bruijn's real-zero time t_+ back to $t = 0$.

Theorem 2.1 (Backward Pick Positivity for the Xi Flow)**.** *Let Ξ be the Riemann Ξ-function and $E_t = e^{-t\partial_z^2}\Xi$ its de Bruijn heat flow (the* Xi Flow*). Assume there exists $t_+ > 0$ such that all zeros of E_{t_+} are real; equivalently,*

$$E_{t_+}(z) \neq 0 \quad \textit{for all } z \in \mathbb{C}_+ \cup \mathbb{C}_- \,. \tag{2.1}$$

(As noted in §1, this hypothesis holds unconditionally by a theorem of de Bruijn [2, 3]; see also Newman [6].)

Then:

(i) For every $t \in [0, t_+]$ one has

$$\Im\big(- E_t'(z)/E_t(z)\big) \;\geq 0 \qquad (z \in \mathbb{C}_+).$$

In particular, at $t = 0$,

$$\Im\big(- \Xi'(z)/\Xi(z)\big) \;\geq 0 \qquad (z \in \mathbb{C}_+).$$

(ii) By Lemma 9.1, any meromorphic function on $\mathbb{C}_+$ with nonnegative imaginary part cannot have poles there. Applying this to $F(z) = -\Xi'(z)/\Xi(z)$, the inequality in (i) *at $t = 0$ implies that $-\Xi'/\Xi$ has no poles in $\mathbb{C}_+$, so Ξ has no zeros in $\mathbb{C}_+$. Since $\Xi(\overline{z}) = \overline{\Xi(z)}$ and Ξ is even, zeros occur in conjugate pairs, and the absence of zeros in $\mathbb{C}_+$ implies there are no zeros in $\mathbb{C}_-$ either. Hence all zeros of Ξ lie on the real axis, so all nontrivial zeros of ζ lie on the critical line.*

(iii) Combined with the Rodgers–Tao lower bound $\Lambda \geq 0$ for the de Bruijn–Newman constant [8], the backward positivity barrier from (i) *forces $\Lambda \leq 0$, hence $\Lambda = 0$, and therefore the Riemann Hypothesis.*

3 Proof Summary and Mechanism

We summarize the six-stage mechanism that leads to theorem 2.1. Each stage maps directly to a section with the same name, and we indicate along the way which lemmas and appendices carry the nonstandard analytic inputs.

Stage 1: Flow, PDE identities, and coarse bounds (Section 4)

We write

$$g(t,z) = -\frac{E_t'(z)}{E_t(z)} = p(x,y,t) + i\,h(x,y,t) \qquad (z = x+iy \in \mathbb{C}_+).$$

On pole-free regions we derive the complex Burgers-type PDE

$$g_t = -g_{zz} + 2g\,g_z,$$

and, since g is holomorphic in z, we have $g_z = g_x$, $g_{zz} = g_{xx}$, so

$$g_t = -g_{xx} + 2g\,g_x.$$

Separating real and imaginary parts, we obtain

$$p_t = -p_{xx} + 2(pp_x - hh_x), \qquad h_t = -h_{xx} + 2(p\,h_x + h\,p_x),$$

and hence the divergence form

$$(\partial_t + \partial_x^2)h = \partial_x(2p\,h), \tag{3.1}$$

see Lemma 4.1. Note that derivatives in y enter only through the Cauchy–Riemann relations, so no y-second derivative appears in the barrier argument.

We assume a Cartwright-type growth bound for Ξ and transfer it to E_t on any compact time interval J via the Gaussian semigroup (lemma 4.2). This yields uniform Cartwright growth (U1) and rectangle zero-counting (U2) for E_t, recorded in (4.3)–(4.4) and proved in Appendix B. These give slab Cauchy bounds for p, p_x away from zeros and Jensen-type control of zero locations.

On a fixed box $\Omega_{Y_0,2R}$ and a collision-free time window $I = [\underline{t}, \overline{t}]$, we show that only finitely many zero branches enter $\Omega_{Y_0,2R}$ and that their mutual separation has a positive lower bound; collision times are discrete (Lemma 4.5). Using the canonical product and Cauchy estimates for E_{zz}/E_z on discs of radius comparable to this separation, we obtain a uniform local speed bound

$$\sup_{t\in I} \sup_{\rho_k(t)\in\Omega_{Y_0,2R}} |\dot\rho_k(t)| \;\le\; L_I < \infty,$$

see Lemma 4.7. This L_I controls how fast the zero set can move along I.

At the base slice t_+, we assume de Bruijn's real-zero property (2.1). Evenness implies that the canonical product for E_{t_+} has no linear exponential factor (Lemma 4.6), so we may write

$$-\frac{E_{t_+}'(z)}{E_{t_+}(z)} = \sum_{\rho\in\mathbb{R}} \frac{1}{\rho - z} + c,$$

with $c \in \mathbb{R}$. For $z = x+iy$ with $y > 0$, each term has positive imaginary part $y/((x-\rho)^2 + y^2)$, so Lemma 4.8 gives

$$\Im\big(-E_{t_+}'/E_{t_+}\big)(z) > 0 \qquad (z \in \mathbb{C}_+),$$

which serves as the initial positivity data for the backward barrier.

Nonstandard inputs in this stage: the Cartwright growth and counting bounds (U1/U2) for E_t (Appendix B), collision discreteness and the local speed bound L_I (Lemmas 4.5 and 4.7), and base-slice positivity from real zeros and evenness (Lemma 4.8).

Stage 2: Time–local algebraic tubes and derivative bounds (Section 5)

On a collision-free window $I = [\underline{t}, \overline{t}]$ we construct a smoothed distance to the instantaneous zero set

$$\delta(t,z) := (z, Z_I(t)), \qquad \delta_\varepsilon(t,\cdot) := \delta(t,\cdot) * \eta_\varepsilon, \quad \eta_\varepsilon(w) = \varepsilon^{-2}\eta(w/\varepsilon),$$

with $\eta \in C_c^\infty(B(0,1))$, $\int \eta = 1$. Lemma 5.2, using the local speed bound L_I, gives

$$\|\nabla \delta_\varepsilon\|_\infty \le 1, \qquad \|\nabla^2 \delta_\varepsilon\|_\infty \lesssim \varepsilon^{-1}, \qquad |\partial_t \delta_\varepsilon| \le L_I.$$

We then define the *time–local algebraic tube*

$$\Theta_{\rho,\varepsilon,\gamma}(t,z) = \vartheta(\delta_\varepsilon(t,z)/\rho) \asymp \Big(\frac{\delta_\varepsilon(t,z)}{\rho}\Big)^\gamma \quad (\delta_\varepsilon \le \rho),$$

with $\gamma > 2$ and $\varepsilon = \theta\rho$. The profile ϑ is smooth, equals s^γ for $s \le 1$, and 1 for $s \ge 2$. Thus Θ vanishes algebraically at the zero set while staying strictly positive, and it localizes to a parabolic tube of radius ρ around $Z_I(t)$.

Writing r for the (smoothed) distance to the nearest zero, Lemma 5.3 gives the *relative-derivative bounds*

$$\frac{|\partial_x \Theta|}{\Theta} \lesssim \frac{\gamma}{r}, \qquad \frac{|\partial_x^2 \Theta|}{\Theta} \lesssim \frac{\gamma(1+\gamma)}{r^2} + \frac{\gamma}{\varepsilon r} + \frac{1}{\rho^2}, \qquad \frac{|\partial_t \Theta|}{\Theta} \lesssim \frac{\gamma L_I}{r}.$$

These are the key inputs in the commutator analysis of the barrier: we never divide by a vanishing weight, only by Θ itself.

We also need a precise description of the drift singularities near zeros. Lemma 5.4 shows that on the tube region $\{r \le 2\rho\}$,

$$|p(z,t)| \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad |p_x(z,t)| \lesssim \frac{1}{r^2} + \frac{1}{\rho^2}.$$

The proof splits the canonical product sum into the nearest zero, finitely many neighbors in a fixed disc (controlled by U2), and a dyadic tail whose contributions are absolutely convergent with $O(1)$ bounds.

Finally, Lemma 5.5 provides *tube integral inequalities* of the form

$$\int_{r\le 2\rho} \frac{F}{r^2} \lesssim \rho^{-2} \int_{r \le 2\rho} F, \qquad \int_{r\le 2\rho} \frac{F}{r^4} \lesssim \rho^{-4} \int_{r \le 2\rho} F,$$

for any nonnegative integrable F. Thus all r^{-2} and r^{-4} singularities on the tube can be traded for ρ^{-2} and ρ^{-4} at the cost of the same w^2 factor. In particular, integrals involving $r^{2\gamma-3}$ and $r^{2\gamma-5}$ converge for $\gamma > 1$; we take $\gamma > 2$ to comfortably control both W_{xx} and $(h^-)_x$.

Nonstandard inputs in this stage: the convolution-smoothing of the distance with explicit ε-dependence (Lemma 5.2), the relative-derivative bounds for Θ (Lemma 5.3), the local singular profile of p, p_x (Lemma 5.4), and the tube integral inequalities (Lemma 5.5).

Stage 3: Backward Carleman–Kato barrier on a short window (Section 6)

On a short time window $[t_1 - \Delta, t_1]$ (contained in a collision-free component I), we introduce two thin time shoulders of thickness τ where $\eta(t)$ decays smoothly to 0 at both endpoints. We set

$$W(t,z) = \eta(t)\,\chi(y)\,y^\alpha\,\Theta_{\rho,\varepsilon,\gamma}(t,z)\,\omega_R(x), \qquad w = W\,h^-, \quad \alpha > 2,$$

with cutoffs χ in y and ω_R in x as in Section 6, and with the lower y–cutoff comparable to ρ on each window.

We apply a one-dimensional backward Carleman identity in x for the operator

$$L := \partial_t + \partial_x^2,$$

with time-singular weight $\phi(t) = \lambda/(t_1 - t)$ blowing up at t_1. In bilinear form (Lemma 6.1) this gives

$$\int (t_1 - t)(|\partial_x w|^2 + \phi'(t)w^2)e^{2\phi} = -\int (t_1 - t)\, Lw\, w\, e^{2\phi} + \frac{1}{2}\int w^2 e^{2\phi},$$

and in particular a Carleman-type control of the positive bulk $\int (t_1 - t)(|\partial_x w|^2 + \phi' w^2)e^{2\phi}$ in terms of the interaction of Lw with w.

On the other hand, on zero-free regions (3.1) gives

$$(\partial_t + \partial_x^2)h = \partial_x(2ph),$$

and the Kato inequality with divergence-form drift (Lemma B.1) yields a *weak Kato decomposition*

$$(\partial_t + \partial_x^2)h^- - \partial_x(2p\, h^-) = \mu \geq 0$$

in the sense of distributions on $\mathbb{R}_x$ for each fixed $y > 0$. Equivalently, h^- is a distributional supersolution of the drifted backward operator. When this is inserted into the Carleman identity with $w = Wh^-$, the defect measure μ contributes only a nonpositive term after multiplication by $(t_1 - t)W^2 e^{2\phi}$, and can be discarded in the ensuing estimates.

Multiplying the weak Kato inequality by $W^2 e^{2\phi}$ and integrating by parts in x splits all drift contributions into terms involving p times either $\partial_x h^-$ or relative derivatives of W. On the zero-free plateau $\mathcal{A}_\tau = \{\Theta \geq \tau\}$, slab Cauchy bounds coming from U1/U2 and standard Cauchy estimates in vertical strips control p, p_x uniformly, and there is a strictly positive zero-free radius, so no singular behavior arises. On the tube $\mathcal{B}_\tau = \{\Theta < \tau\}$, we express all commutators using relative derivatives

$$\frac{W_x}{W}, \quad \frac{W_{xx}}{W}, \quad \frac{\partial_t W}{W},$$

which are bounded in terms of r, ρ, ε by Lemma 5.3; and we bound the drift p, p_x by Lemma 5.4. The tube integrals with r^{-2} and r^{-4} weights are converted into ρ^{-2} and ρ^{-4} factors via Lemma 5.5, at the cost of the same w^2 factor. In all cases, the right-hand side of the Carleman identity becomes a sum of terms of the form

$$\int (t_1 - t)\,(\rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2})\, w^2 e^{2\phi},$$

which we compare against the positive Carleman bulk $\int (t_1 - t)\phi' w^2 e^{2\phi}$.

We choose the *cubic coupling*

$$\rho = \lambda^{-1/2}, \qquad \Delta \;\lesssim\; \min\{\lambda^{-1/3},\, L_I^{-2/3}\},$$

so that for instance

$$\frac{(t_1 - t)\,\rho^{-4}}{\phi'(t)} \;\sim\; \frac{(t_1 - t)\,\lambda^2}{\lambda/(t_1 - t)^2} \;=\; (t_1 - t)^3\,\lambda,$$

and $(t_1 - t)^3 \lesssim \lambda^{-1}$ guarantees absorption. A similar calculation shows that the L_I-dependent terms are absorbed under $\Delta^3 L_I^2 \lesssim 1$, i.e. under $\Delta \lesssim L_I^{-2/3}$. On each window we choose the lower y–cutoff comparable to ρ, so the y-dependent pieces in p, p_x scale compatibly with ϕ' and are absorbed in the same regime. The thin time shoulders contribute controlled error terms proportional to Δ^{-1}, which are likewise absorbed once λ is large. All of this is combined in Proposition 6.3, which yields backward positivity of h on the window $[t_1 - \Delta, t_1]$.

Nonstandard inputs in this stage: the backward Carleman identity with blow-up weight (Lemma 6.1), the Kato inequality with drift in divergence form and nonnegative defect (Lemma B.1 and its proof in Appendix B), the slab bounds for p, p_x obtained from U1/U2 and Cauchy estimates, and the explicit absorption under the cubic coupling in Proposition 6.3 using the tube machinery of Stage 2.

Stage 4: Collision bridging (Section 7)

On a compact interval we have only finitely many collision times by Lemma 4.5 (see also Angenent [1]). To propagate positivity across them, we introduce a tube-weighted energy

$$\mathcal{E}(t) = \iint \left((h^-)^2 + |(h^-)_x|^2\right) W(t,z)^2 \, dx \, dy,$$

with W as in Stage 3. The algebraic vanishing of Θ near the zeros, the y^α weight with $\alpha > 2$, and the Cartwright bounds ensure that all terms in $\mathcal{E}(t)$ are integrable and that $\mathcal{E}$ is absolutely continuous up to any collision time; this is made precise in Lemma 7.1. In particular, $\mathcal{E}$ has finite one-sided limits at a collision time t^*.

If $h \geq 0$ on $\mathbb{C}_+$ for $t \in [t^*, t^* + \delta]$ (coming from a window above the collision), then $\mathcal{E}(t^*) = 0$. For any $t_0 < t^*$ sufficiently close to t^*, continuity of $\mathcal{E}$ gives $\mathcal{E}(t_0)$ arbitrarily small. We then apply the short-time barrier of Stage 3, starting from t_0 and going backward on a small collision-free window below t_0, to force $h^- = 0$ there. Lemma 7.3 formalizes this "restart" mechanism, and iterating across the finitely many collisions in a compact time interval yields backward positivity on the entire interval.

Stage 5: Exhaustion in (x, y) and global backward positivity (Section 8)

The barrier of Stage 3 is proved with compactly supported cutoffs $\omega_R(x)$ and $\chi(y)$. To extend from the truncated domains to all of $\mathbb{C}_+$ we let $R \to \infty$ and the lower y–cutoff $\rho \downarrow 0$ along a diagonal sequence. The small-y behavior is controlled by the y^α factor with $\alpha > 2$, which makes $\int_0^1 y^{2\alpha} \, dy$ finite and ensures that the weighted energy captures any negative part near $y = 0$. The large-y behavior is controlled by the vertical Cartwright bounds (U1), which give uniform growth control for E_t and E_t' in vertical strips. Lemma 8.1 carries out this exhaustion argument and shows that for each $t \leq t_+$ we have $h^-(\cdot, \cdot, t) \equiv 0$ on all of $\mathbb{C}_+$, that is

$$\Im\big(-E_t'/E_t\big)(z) \ \geq 0 \qquad (z \in \mathbb{C}_+, \ t \leq t_+).$$

Stage 6: Pick positivity and real zeros (Section 9)

At $t = 0$ we conclude

$$\Im\big(-\Xi'(z)/\Xi(z)\big) \ \geq 0 \qquad (z \in \mathbb{C}_+).$$

Lemma 9.1, a simple Pick/Nevanlinna lemma, states that a meromorphic function F on $\mathbb{C}_+$ with $\Im F \geq 0$ cannot have poles in $\mathbb{C}_+$; the proof uses the behavior of the imaginary part on small circles around putative poles. Applying this to $F = -\Xi'/\Xi$ shows that Ξ has no zeros in $\mathbb{C}_+$. By evenness, all zeros of Ξ lie on the real line, so all nontrivial zeros of ζ lie on the critical line.

Combined with the Rodgers–Tao lower bound $\Lambda \geq 0$ for the de Bruijn–Newman constant, and the fact that our backward barrier shows Ξ has only real zeros already at $t = 0$, we obtain $\Lambda = 0$, and hence the Riemann Hypothesis.

Taken together, these six stages implement a parabolic positivity mechanism for RH: starting from de Bruijn's real-zero slice at t_+, the backward Carleman–Kato barrier with time–local algebraic tubes propagates $\Im(-E_t'/E_t) \geq 0$ down to $t = 0$, and a final Pick-positivity step upgrades this to real zeros of Ξ and $\Lambda = 0$.

3.1 Dependency diagram

Figure 1 shows the logical flow of the proof of Theorem 2.1 as a directed acyclic graph. Each arrow indicates that results in one stage are used in the proofs of another. The bottom row contains the external inputs of de Bruijn and Rodgers–Tao.

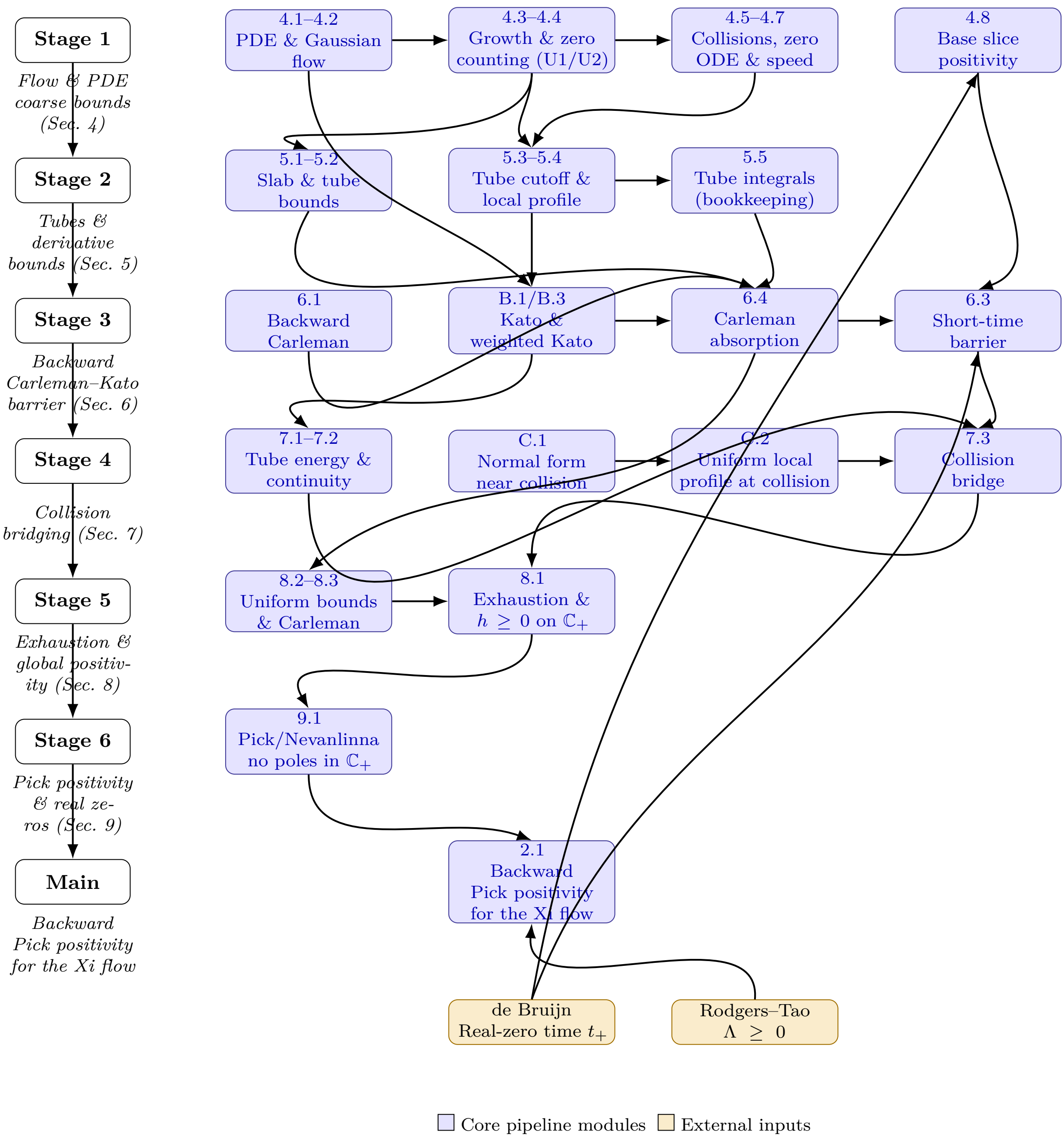

Figure 1: Dependency diagram for the Xi flow proof of Theorem 2.1 (Backward Pick positivity for the Xi Flow). Violet boxes: core modules in the flow argument. Orange boxes: external classical inputs (de Bruijn real-zero slice and Rodgers–Tao bound).

4 Stage 1: Flow, PDE identities, and coarse bounds

Throughout we work with $E_t = e^{-t\partial_z^2}\Xi$ (for background on ζ and Ξ, see Edwards [4]) and assume the existence of a real-zero slice $t_+ > 0$ as in (1.1). The fact that such a t_+ exists is due to de Bruijn [2, 3]; we use only the real-zero property and not any further structure (such as simplicity of the zeros).

Let $E_t = e^{-t\partial_z^2}\Xi$ satisfy $\partial_t E_t = -\partial_z^2 E_t$. Set

$$g(t,z) := -\frac{E_t'(z)}{E_t(z)} = p(x,y,t) + i\,h(x,y,t) \qquad (z = x+iy \in \mathbb{C}_+).$$

Lemma 4.1 (PDE for p, h)**.** *On any open set where E_t has no zeros, g satisfies*

$$g_t = -g_{zz} + 2gg_z,$$

and writing $g = p + ih$ with $z = x + iy$,

$$p_t = -p_{xx} + 2(pp_x - hh_x), \qquad h_t = -h_{xx} + 2(p\,h_x + h\,p_x),$$

in particular

$$(\partial_t + \partial_x^2)h = \partial_x(2p\,h). \tag{4.1}$$

Proof. Differentiate the definition $g = -E_z/E$ with respect to t:

$$g_t = -\frac{E_{zt}}{E} + \frac{E_z E_t}{E^2}.$$

Using $E_t = -E_{zz}$ (heat flow) we get $E_{zt} = -E_{zzz}$ and $E_t = -E_{zz}$, whence

$$g_t = \frac{E_{zzz}}{E} - \frac{E_z E_{zz}}{E^2}.$$

On the other hand,

$$g_z = -\frac{E_{zz}}{E} + \frac{E_z^2}{E^2}, \qquad g_{zz} = -\frac{E_{zzz}}{E} + 3\frac{E_z E_{zz}}{E^2} - 2\frac{E_z^3}{E^3}.$$

Thus

$$\begin{aligned} -g_{zz} + 2gg_z &= \frac{E_{zzz}}{E} - 3\frac{E_z E_{zz}}{E^2} + 2\frac{E_z^3}{E^3} + 2\Big(-\frac{E_z}{E}\Big)\Big(-\frac{E_{zz}}{E} + \frac{E_z^2}{E^2}\Big) \\ &= \frac{E_{zzz}}{E} - 3\frac{E_z E_{zz}}{E^2} + 2\frac{E_z^3}{E^3} + 2\frac{E_z E_{zz}}{E^2} - 2\frac{E_z^3}{E^3} \\ &= \frac{E_{zzz}}{E} - \frac{E_z E_{zz}}{E^2} = g_t. \end{aligned}$$

Since g is holomorphic in z, we have $g_z = g_x$ and $g_{zz} = g_{xx}$, hence

$$g_t = -g_{xx} + 2gg_x.$$

Writing $g = p + ih$ and $g_x = p_x + ih_x$ gives

$$p_t + ih_t = -(p_{xx} + ih_{xx}) + 2(p + ih)(p_x + ih_x),$$

and comparing real and imaginary parts yields

$$p_t = -p_{xx} + 2(pp_x - hh_x), \qquad h_t = -h_{xx} + 2(p\,h_x + h\,p_x).$$

Finally,

$$(\partial_t + \partial_x^2)h = h_t + h_{xx} = 2(p\,h_x + h\,p_x) = \partial_x(2ph).$$

□

Lemma 4.2 (Gaussian semigroup)**.** *For entire f and $t > 0$,*

$$(e^{-t\partial_z^2} f)(z) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)} f(z + is)\, ds.$$

Proof. Fix $z \in \mathbb{C}$ and set

$$k_t(s) = \frac{1}{\sqrt{4\pi t}} e^{-s^2/(4t)}, \qquad u(t,z) = \int_{\mathbb{R}} k_t(s)\, f(z + is)\, ds.$$

Since f is entire of order at most 1 (Cartwright hypothesis), it is of at most exponential growth in vertical strips, while k_t is Gaussian, so the integral is absolutely convergent and defines an entire function of z for each $t > 0$, smooth in t.

The kernel satisfies the one-dimensional heat equation

$$\partial_t k_t(s) = \partial_s^2 k_t(s),$$

hence

$$\partial_t u(t,z) = \int_{\mathbb{R}} \partial_t k_t(s)\, f(z + is)\, ds = \int_{\mathbb{R}} \partial_s^2 k_t(s)\, f(z + is)\, ds.$$

Integrating by parts twice in s (the boundary terms vanish by Gaussian decay) gives

$$\partial_t u(t,z) = \int_{\mathbb{R}} k_t(s)\, \partial_s^2 f(z + is)\, ds.$$

Since $\partial_s f(z + is) = i f'(z + is)$, we have $\partial_s^2 f(z + is) = -f''(z + is)$, so

$$\partial_t u(t,z) = -\int_{\mathbb{R}} k_t(s)\, f''(z + is)\, ds = -\partial_z^2 u(t,z).$$

As $t \downarrow 0$, the Gaussian k_t is an approximate identity on $\mathbb{R}$ and by dominated convergence we have $u(t,z) \to f(z)$. Hence u solves

$$\partial_t u = -\partial_z^2 u, \qquad u(0,\cdot) = f,$$

and by uniqueness of the heat flow with entire initial datum, $u(t,z) = (e^{-t\partial_z^2} f)(z)$. $\square$

Cartwright baseline (U1/U2). Assume the coarse bound

$$\log|\Xi(w)| \;\le\; A_\Xi(1 + |w|)\log(2 + |w|) + B_\Xi \qquad (\forall w \in \mathbb{C}). \tag{4.2}$$

Then, for any compact $J = [t_1, t_2] \Subset (0,\infty)$, there are explicit constants $C_J, K_J, \widetilde{C}_J$ with:

$$|E_t(z)| \;\le\; C_J \exp\big(A_\Xi(1 + |z|)\log(2 + |z|)\big), \quad t \in J, \tag{4.3}$$

$$N_t(R;H) := \#\{\rho(t) : |\Re\rho| \le R, |\Im\rho| \le H\} \;\le\; K_J + \widetilde{C}_J (R + H)\log(2 + R + H). \tag{4.4}$$

Lemma 4.3 (U1: uniform Cartwright growth)**.** *Under (4.2), for any compact $J \Subset (0,\infty)$ there exists C_J such that (4.3) holds for all $t \in J$.*

Proof. By lemma 4.2,

$$E_t(z) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)}\, \Xi(z + is)\, ds.$$

Using (4.2) with $w = z + is$ yields

$$|\Xi(z + is)| \le \exp\big(A_\Xi(1 + |z + is|)\log(2 + |z + is|) + B_\Xi\big).$$

Since $|z+is| \le |z|+|s|$ and $\log(2+|z+is|) \le \log(2+|z|+|s|)$,

$$|\Xi(z+is)| \le C_0 \exp\big(A_\Xi(|z|+|s|)\log(2+|z|+|s|)\big),$$

for some C_0 depending on A_Ξ, B_Ξ. On any compact J we have $t \in [t_1, t_2]$ and $t^{-1/2} \le c_J$, so

$$|E_t(z)| \le C_1 \int_{\mathbb{R}} \exp\big(-s^2/(4t_2) + A_\Xi(|z|+|s|)\log(2+|z|+|s|)\big)\,ds.$$

For fixed z, the integrand is dominated by a Gaussian times a stretched exponential in s, which is integrable; the integral is thus bounded by

$$C_J \exp\big(A_\Xi(1+|z|)\log(2+|z|)\big)$$

for some C_J depending on J and A_Ξ, B_Ξ. This proves (4.3). $\square$

Lemma 4.4 (U2: rectangle zero counting)**.** *Under* (4.2)*, for any compact $J \Subset (0,\infty)$ there exist $K_J, \widetilde{C}_J$ such that* (4.4) *holds for all $t \in J$.*

Proof. Fix $t \in J$ and consider E_t as an entire function. By lemma 4.3, E_t has order 1 and type at most A_Ξ uniformly in $t \in J$. Jensen's formula in discs (or the standard Cartwright theory, see e.g. Levin [5], *Distribution of Zeros of Entire Functions*) implies that the number of zeros in $\{|z| \le R\}$ satisfies

$$N_t(R) \ \le\ K'_J + \widetilde{C}'_J R\log(2+R).$$

Replacing discs by rectangles $\{|\Re z| \le R, |\Im z| \le H\}$ enlarges the domain by a constant factor, and the Jensen integral over a rectangle is bounded by that over a disc of comparable radius. Thus we obtain

$$N_t(R;H) \ \le\ K_J + \widetilde{C}_J(R+H)\log(2+R+H)$$

with constants independent of $t \in J$. $\square$

Lemma 4.5 (Collision discreteness in a bounded region)**.** *Let $J \Subset (0,\infty)$ be compact and let $\Omega \subset \mathbb{C}$ be bounded. Then the set*

$$\{t \in J : \exists z \in \Omega,\ E_t(z) = E_z(t,z) = 0\}$$

is finite.

Proof. For clarity we temporarily write

$$E(t,z) := E_t(z)$$

so that $\partial_t E$ denotes the t–derivative and E_z the z–derivative.

Step 1: Holomorphic extension in (t,z) and the heat equation. Fix a compact interval $J = [t_1, t_2] \Subset (0,\infty)$ and a bounded set $\Omega \subset \mathbb{C}$. Choose $R > 0$ so that $\Omega \subset \{z : |z| < R\}$.

By lemma 4.2, for every real $t > 0$ we have

$$E(t,z) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)}\,\Xi(z+is)\,ds.$$

Let $D_t \subset \{t \in \mathbb{C} : \Re t > 0\}$ be a disc whose real diameter contains J, and let $D_z := \{z \in \mathbb{C} : |z| < R+1\}$. For $t \in D_t$ and $z \in D_z$ define

$$\widetilde{E}(t,z) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)}\,\Xi(z+is)\,ds,$$

where we use the principal branch of $\sqrt{t}$.

Since Ξ is of at most exponential growth in vertical strips (Cartwright assumption (4.2)) and for $\Re t > 0$ we have

$$\Re\Big(\frac{1}{t}\Big) = \frac{\Re t}{|t|^2} > 0,$$

the Gaussian factor $e^{-s^2/(4t)}$ decays like $e^{-c|s|^2}$ for some $c > 0$ depending only on D_t. Thus the integral converges absolutely and uniformly on compact subsets of $D_t \times D_z$, and standard dominated convergence shows that $\widetilde{E}$ is holomorphic in (t, z) on

$$U := D_t \times D_z.$$

For real $t \in J$ we have $\widetilde{E}(t, z) = E(t, z)$ by lemma 4.2, so we simply regard E as the holomorphic extension $\widetilde{E}$ on U.

The same argument differentiating under the integral sign shows that E_z and E_{zz} are holomorphic on U, and for real $t > 0$ the heat equation

$$\partial_t E(t, z) = -\partial_z^2 E(t, z)$$

holds. Since both sides are holomorphic functions of t on D_t (for each fixed $z \in D_z$) and agree for $t \in J$ with an accumulation point, the identity extends to all $t \in D_t$ by analytic continuation. Thus on U we have

$$\partial_t E(t, z) + \partial_z^2 E(t, z) = 0. \tag{4.5}$$

Step 2: The multiple-zero set is a 1–dimensional analytic set with no vertical slab. Define

$$\Sigma := \{(t, z) \in U : \ E(t, z) = 0, \ E_z(t, z) = 0\}.$$

Since E and E_z are holomorphic on U, Σ is the common zero set of the holomorphic map

$$(E, E_z) : U \to \mathbb{C}^2,$$

and hence a complex analytic subset of U. Because E is not identically zero (on $t = 0$ we recover Ξ, which is nontrivial), Σ cannot contain a nonempty open subset of U, so $\dim_{\mathbb{C}} \Sigma \le 1$.

We claim that Σ has no *vertical slab*, i.e. there is no nonempty disc $D \subset D_t$ and point $z_0 \in D_z$ with

$$D \times \{z_0\} \subset \Sigma. \tag{4.6}$$

Assume for contradiction that (4.6) holds. Then for all $t \in D$,

$$E(t, z_0) = 0, \qquad E_z(t, z_0) = 0.$$

Set $G(t, w) := E(t, z_0 + w)$. Then G is holomorphic on some polydisc $D \times \{|w| < \varepsilon\} \subset U$, and by (4.5) we have

$$G_t(t, w) + G_{ww}(t, w) = E_t(t, z_0 + w) + E_{zz}(t, z_0 + w) = 0.$$

At $w = 0$,

$$G(t, 0) = E(t, z_0) = 0, \qquad G_w(t, 0) = E_z(t, z_0) = 0 \qquad (t \in D).$$

Thus $w = 0$ is a zero of multiplicity at least 2 of the entire function $w \mapsto G(t, w)$ for every $t \in D$.

Expand G in a Taylor series in w:

$$G(t, w) = \sum_{k \ge m} c_k(t) w^k, \qquad m \ge 2,$$

with $c_k(t)$ holomorphic on D. The conditions $G(t,0)=0$ and $G_w(t,0)=0$ imply

$$c_0(t) \equiv 0, \qquad c_1(t) \equiv 0.$$

We also have

$$G_t(t,w) = \sum_{k\geq m} c_k'(t)w^k, \qquad G_{ww}(t,w) = \sum_{k\geq m} k(k-1)c_k(t)w^{k-2},$$

so $G_t + G_{ww} = 0$ becomes

$$\sum_{k\geq m} c_k'(t)w^k + \sum_{k\geq m} k(k-1)c_k(t)w^{k-2} = 0.$$

Reindexing the second sum by $\ell = k-2$ yields, for all $\ell \geq 0$,

$$c_\ell'(t) + (\ell+2)(\ell+1)c_{\ell+2}(t) = 0,$$

where we interpret $c_k \equiv 0$ for $k < m$.

For $\ell = 0$, using $c_0 \equiv 0$ we obtain $2c_2(t) = 0$, so $c_2 \equiv 0$. For $\ell = 1$, using $c_1 \equiv 0$ we obtain $6c_3(t) = 0$, so $c_3 \equiv 0$. Assume inductively that $c_0, \ldots, c_{\ell+1} \equiv 0$ for some $\ell \geq 1$. Then the equation for index ℓ becomes

$$(\ell+2)(\ell+1)c_{\ell+2}(t) = 0,$$

so $c_{\ell+2} \equiv 0$. Hence all coefficients c_k vanish identically on D, and therefore $G \equiv 0$ on $D \times \{|w| < \varepsilon\}$.

Undoing the translation $w = z - z_0$, we find that E vanishes identically in a neighbourhood of $D \times \{z_0\}$ in U, and by analytic continuation $E \equiv 0$ on U, contradicting the definition of E. Thus Σ has no vertical slab.

Step 3: Local discreteness of the t–projection. We now show that for every point $(t_0, z_0) \in U$ there exists a polydisc $V \subset U$ centred at (t_0, z_0) such that the projection

$$\pi_t(\Sigma \cap V) \subset \mathbb{C}_t$$

is a discrete subset of $\pi_t(V)$. Once this is established, the lemma follows: since $J \times \Omega$ is compact and contained in U, we can cover it by finitely many such polydiscs $V_1, \ldots, V_M$; for each j, $\pi_t(\Sigma \cap V_j) \cap J$ is discrete in the compact set J and hence finite, and

$$\{t \in J : \exists z \in \Omega,\ (t,z) \in \Sigma\} \subset \bigcup_{j=1}^{M} \left(\pi_t(\Sigma \cap V_j) \cap J\right)$$

is therefore finite.

The only nontrivial case is when $(t_0, z_0) \in \Sigma$, i.e.

$$E(t_0, z_0) = 0, \qquad E_z(t_0, z_0) = 0.$$

By translating coordinates we may assume $(t_0, z_0) = (0,0)$ and that U is a small polydisc $\Delta_t \times \Delta_z$ around $(0,0)$.

Because $E(0,\cdot)$ has a zero of multiplicity at least 2 at $z = 0$, the Weierstrass preparation theorem applied in the z–variable gives:

- an integer $m \geq 2$ (the multiplicity at $(0,0)$),

- a holomorphic, nowhere-vanishing function $H(t,z)$ on U, and
- a monic polynomial

$$W(t,z) = z^m + a_{m-1}(t)z^{m-1} + \cdots + a_0(t),$$

  with $a_j(t)$ holomorphic on Δ_t,

such that

$$E(t,z) = W(t,z)\,H(t,z) \qquad ((t,z) \in U). \tag{4.7}$$

In particular, the zeros of $E(t,\cdot)$ in Δ_z are exactly the zeros of $W(t,\cdot)$.

At a point $(t,z) \in U$ with $E(t,z) = 0$ we have $W(t,z) = 0$ and $H(t,z) \neq 0$, and differentiating (4.7) in z gives

$$E_z(t,z) = W_z(t,z)H(t,z) + W(t,z)H_z(t,z),$$

so on $\{W = 0\}$ we have $E_z(t,z) = 0$ if and only if $W_z(t,z) = 0$. Thus, in a neighbourhood of $(0,0)$,

$$\Sigma = \{(t,z) \in U : \; W(t,z) = 0, \; W_z(t,z) = 0\}.$$

For each fixed $t \in \Delta_t$, write $W_t(z) := W(t,z)$; this is a monic degree m polynomial in z. Recall that W_t has a multiple root if and only if its discriminant $\Delta_W(t)$ vanishes. The discriminant $\Delta_W(t)$ is a (polynomial) expression in the coefficients $a_j(t)$, hence defines a holomorphic function

$$\Delta_W : \Delta_t \longrightarrow \mathbb{C}.$$

For $t \in \Delta_t$, the equation $W(t,z) = W_z(t,z) = 0$ has a solution $z \in \Delta_z$ if and only if W_t has a multiple root in Δ_z, i.e. if and only if $\Delta_W(t) = 0$. Therefore

$$\pi_t(\Sigma \cap U) \subset \{t \in \Delta_t : \Delta_W(t) = 0\}.$$

If Δ_W is not identically zero, then its zero set $\{t \in \Delta_t : \Delta_W(t) = 0\}$ is a discrete subset of Δ_t, and we can take $V = U$, which has the desired property.

It remains to rule out the case $\Delta_W \equiv 0$ on Δ_t. In that case, for every $t \in \Delta_t$, the polynomial W_t has a multiple root, so $\Sigma \cap U$ meets every vertical slice $\{t\} \times \Delta_z$; in particular, the projection $\pi_t(\Sigma \cap U)$ is all of Δ_t.

Since $\Sigma \cap U$ is a 1–dimensional analytic subset of U, it admits a decomposition into finitely many irreducible 1–dimensional components and isolated points. Because $\pi_t(\Sigma \cap U) = \Delta_t$ is not discrete, there exists an irreducible component $C \subset \Sigma \cap U$ whose projection $\pi_t(C)$ contains a nonempty disc $D \subset \Delta_t$. By the local structure of 1–dimensional analytic sets in $\mathbb{C}^2$, there is a holomorphic function $a : D \to \Delta_z$ such that

$$(t, a(t)) \in C \subset \Sigma \qquad \text{for all } t \in D.$$

Equivalently,

$$E(t,a(t)) = 0, \qquad E_z(t,a(t)) = 0 \qquad (t \in D). \tag{4.8}$$

Step 4: A moving multiple-zero curve forces $E \equiv 0$. Define

$$G(t,w) := E(t, a(t) + w).$$

As E is holomorphic on U and a is holomorphic on D, the function G is holomorphic on $D \times \{|w| < \varepsilon\}$ for some small $\varepsilon > 0$.

Differentiating and using (4.5) we obtain

$$G_t(t,w) = E_t(t,a(t)+w) + a'(t)E_z(t,a(t)+w),$$

$$G_w(t,w) = E_z(t,a(t)+w), \qquad G_{ww}(t,w) = E_{zz}(t,a(t)+w),$$

hence

$$G_t + G_{ww} - a'(t)G_w = [E_t + E_{zz}](t,a(t)+w) = 0.$$

Thus G satisfies the convection–diffusion equation

$$G_t + G_{ww} - a'(t)G_w = 0 \qquad \text{on } D \times \{|w| < \varepsilon\}. \tag{4.9}$$

From (4.8) we have, for all $t \in D$,

$$G(t,0) = E(t,a(t)) = 0, \qquad G_w(t,0) = E_z(t,a(t)) = 0,$$

so $w = 0$ is a zero of multiplicity at least 2 of $G(t,\cdot)$ for every $t \in D$.

Expand

$$G(t,w) = \sum_{k \geq m} c_k(t)w^k, \qquad m \geq 2,$$

with $c_k(t)$ holomorphic on D. Then $G(t,0) = 0$ and $G_w(t,0) = 0$ imply $c_0 \equiv 0$ and $c_1 \equiv 0$. From (4.9) we compute

$$G_t(t,w) = \sum_{k \geq m} c_k'(t)w^k, \qquad G_w(t,w) = \sum_{k \geq m} kc_k(t)w^{k-1}, \qquad G_{ww}(t,w) = \sum_{k \geq m} k(k-1)c_k(t)w^{k-2},$$

so

$$\sum_{k \geq m} c_k'(t)w^k + \sum_{k \geq m} k(k-1)c_k(t)w^{k-2} - a'(t)\sum_{k \geq m} kc_k(t)w^{k-1} = 0.$$

Reindexing and equating coefficients of w^ℓ for $\ell \geq 0$ yields

$$c_\ell'(t) + (\ell+2)(\ell+1)c_{\ell+2}(t) - a'(t)(\ell+1)c_{\ell+1}(t) = 0.$$

For $\ell = 0$, using $c_0 \equiv 0$ and $c_1 \equiv 0$ we get

$$0 = c_0'(t) + 2c_2(t) - a'(t)c_1(t) = 2c_2(t),$$

so $c_2 \equiv 0$. For $\ell = 1$, using $c_1 \equiv 0$ and $c_2 \equiv 0$,

$$0 = c_1'(t) + 6c_3(t) - 2a'(t)c_2(t) = 6c_3(t),$$

so $c_3 \equiv 0$. Inductively, suppose $c_0, \ldots, c_{\ell+1} \equiv 0$ for some $\ell \geq 1$. Then the ℓ–equation becomes

$$0 = c_\ell'(t) + (\ell+2)(\ell+1)c_{\ell+2}(t) - a'(t)(\ell+1)c_{\ell+1}(t) = (\ell+2)(\ell+1)c_{\ell+2}(t),$$

so $c_{\ell+2} \equiv 0$. Thus all c_k vanish identically, and therefore

$$G(t,w) \equiv 0 \quad \text{on } D \times \{|w| < \varepsilon\}.$$

Undoing the change of variables $z = a(t) + w$, we conclude that E vanishes on an open subset of U, hence $E \equiv 0$ on U by analytic continuation, again a contradiction.

This contradiction shows that Δ_W cannot be identically zero, so $\{t \in \Delta_t : \Delta_W(t) = 0\}$ is a discrete subset of Δ_t, and therefore $\pi_t(\Sigma \cap U)$ is discrete. As explained at the beginning of Step 3, this implies that, for any compact $J \Subset (0,\infty)$ and bounded $\Omega \subset \mathbb{C}$, the set of collision times in J for zeros lying in Ω is finite. $\square$

Lemma 4.6 (Canonical product and zero motion)**.** *For each fixed $t > 0$, E_t is an entire function of order 1 and finite type, hence has a genus-1 canonical product*

$$E_t(z) = e^{\beta_t z + \gamma_t} \prod_{\rho(t)} \Big(1 - \frac{z}{\rho(t)}\Big) \exp\Big(\frac{z}{\rho(t)}\Big),$$

where the product is over zeros $\rho(t)$ of E_t. Evenness $E_t(-z) = E_t(z)$ forces $\beta_t = 0$, and the zeros are symmetric: if $\rho(t)$ is a zero then so is $-\rho(t)$ with the same multiplicity.

On any time interval where all zeros in a given bounded region are simple (i.e. collision-free), each such zero $z_j(t) = x_j(t) + iy_j(t)$ depends real-analytically on t and satisfies

$$\dot z_j(t) = 2 \sum_{k \neq j} \frac{1}{z_j(t) - z_k(t)}, \tag{4.10}$$

and

$$\frac{d}{dt}(y_j(t)^2) = -4 \sum_{k \neq j} \frac{y_j(t)(y_j(t) - y_k(t))}{|z_j(t) - z_k(t)|^2}. \tag{4.11}$$

Proof. The canonical product representation with linear exponential factor is standard for entire functions of order 1 and finite type; see e.g. Levin [5]. Evenness $E_t(-z) = E_t(z)$ gives

$$e^{\beta_t z + \gamma_t} \prod_{\rho} \Big(1 - \frac{z}{\rho}\Big) \exp\Big(\frac{z}{\rho}\Big) = e^{-\beta_t z + \gamma_t} \prod_{\rho} \Big(1 + \frac{z}{\rho}\Big) \exp\Big(-\frac{z}{\rho}\Big).$$

Since the zero set is symmetric under $z \mapsto -z$, the product over ρ is even, hence the exponential factor must be even as well: $e^{\beta_t z} = e^{-\beta_t z}$ for all z, forcing $\beta_t = 0$.

Write the multiset of zeros (counted with multiplicity) as $\{\pm\rho_j(t)\}_j$, together with a possible zero at 0 (which plays no role in what follows). Grouping the canonical product into $\pm$–pairs, we obtain

$$\begin{aligned} E_t(z) &= e^{\gamma_t} \prod_j \Big(1 - \frac{z}{\rho_j(t)}\Big) \exp\Big(\frac{z}{\rho_j(t)}\Big) \Big(1 - \frac{z}{-\rho_j(t)}\Big) \exp\Big(\frac{z}{-\rho_j(t)}\Big) \\ &= e^{\gamma_t} \prod_j \Big(1 - \frac{z}{\rho_j(t)}\Big) \Big(1 + \frac{z}{\rho_j(t)}\Big) \exp\Big(\frac{z}{\rho_j(t)} + \frac{z}{-\rho_j(t)}\Big) \\ &= e^{\gamma_t} \prod_j \Big(1 - \frac{z^2}{\rho_j(t)^2}\Big), \end{aligned}$$

since the exponential factors cancel in each pair.

Differentiating $\log E_t(z)$ then gives

$$\frac{E_t'(z)}{E_t(z)} = \sum_j \frac{-2z}{\rho_j(t)^2 - z^2} = \sum_j \Big(\frac{1}{z - \rho_j(t)} + \frac{1}{z + \rho_j(t)}\Big) = \sum_k \frac{1}{z - \rho_k(t)},$$

where the last sum is over all zeros $\rho_k(t)$, positive and negative. This is a locally absolutely convergent sum in z, and we record it as

$$\frac{E_t'(z)}{E_t(z)} = \sum_k \frac{1}{z - \rho_k(t)}. \tag{4.12}$$

Now fix a simple zero $z_j(t)$ in a collision-free interval. Factor

$$E_t(z) = (z - z_j(t))G_t(z),$$

where G_t is entire and $G_t(z_j(t)) \neq 0$. Then

$$E_t'(z) = G_t(z) + (z - z_j(t))G_t'(z),$$

so

$$E_t'(z_j(t)) = G_t(z_j(t)), \qquad E_t''(z_j(t)) = 2G_t'(z_j(t)).$$

View $E_t(z)$ as the restriction in z of a function $E(t, z)$ solving the heat equation

$$\partial_t E(t, z) = -\partial_{zz} E(t, z).$$

Differentiating the identity $E(t, z_j(t)) = 0$ in t then gives

$$0 = \frac{d}{dt} E(t, z_j(t)) = \partial_t E(t, z_j(t)) + \dot{z}_j(t)\, \partial_z E(t, z_j(t)) = -\partial_{zz} E(t, z_j(t)) + \dot{z}_j(t)\, \partial_z E(t, z_j(t)),$$

hence

$$\dot{z}_j(t) = \frac{\partial_{zz} E}{\partial_z E}(t, z_j(t)).$$

At a fixed time t we have $\partial_z E(t, z) = E_t'(z)$ and $\partial_{zz} E(t, z) = E_t''(z)$, so using the identities above,

$$\dot{z}_j(t) = \frac{E_t''(z_j(t))}{E_t'(z_j(t))} = \frac{2G_t'(z_j(t))}{G_t(z_j(t))} = 2\frac{d}{dz} \log G_t(z)\Big|_{z=z_j(t)}.$$

Using (4.12), we have

$$\frac{E_t'(z)}{E_t(z)} = \frac{1}{z - z_j(t)} + \sum_{k \neq j} \frac{1}{z - \rho_k(t)},$$

and therefore

$$\frac{G_t'(z)}{G_t(z)} = \frac{E_t'(z)}{E_t(z)} - \frac{1}{z - z_j(t)} = \sum_{k \neq j} \frac{1}{z - \rho_k(t)}.$$

Evaluating at $z = z_j(t)$ yields

$$\frac{G_t'(z_j(t))}{G_t(z_j(t))} = \sum_{k \neq j} \frac{1}{z_j(t) - \rho_k(t)}.$$

Relabelling the zeros as $z_k(t) := \rho_k(t)$, we obtain

$$\dot{z}_j(t) = 2 \sum_{k \neq j} \frac{1}{z_j(t) - z_k(t)},$$

which proves (4.10).

For $y_j(t) = \Im z_j(t)$, we have

$$\dot{y}_j(t) = \Im \dot{z}_j(t) = 2 \sum_{k \neq j} \Im \frac{1}{z_j(t) - z_k(t)}.$$

Writing $z_j - z_k = \Delta x_{jk} + i\Delta y_{jk}$,

$$\Im \frac{1}{z_j - z_k} = \Im \frac{\Delta x_{jk} - i\Delta y_{jk}}{\Delta x_{jk}^2 + \Delta y_{jk}^2} = -\frac{\Delta y_{jk}}{|z_j - z_k|^2} = -\frac{y_j - y_k}{|z_j - z_k|^2}.$$

Thus

$$\dot{y}_j(t) = -2 \sum_{k \neq j} \frac{y_j(t) - y_k(t)}{|z_j(t) - z_k(t)|^2},$$

and therefore

$$\frac{d}{dt}(y_j(t)^2) = 2y_j(t)\dot{y}_j(t) = -4 \sum_{k \neq j} \frac{y_j(t)(y_j(t) - y_k(t))}{|z_j(t) - z_k(t)|^2},$$

which is (4.11). □

Lemma 4.7 (Local speed bound on a collision-free window)**.** *Let $I = [\underline{t}, \overline{t}] \subset (0, \infty)$ be collision-free in a box $\Omega_{Y_0,2R} = \{(x, y) : |x| \le 2R,\ 0 < y \le Y_0\}$. Then there exists $L_I < \infty$ such that*

$$\sup_{t \in I} \sup_{\rho_k(t) \in \Omega_{Y_0,2R}} |\dot{\rho}_k(t)| \ \le \ L_I.$$

Proof. By lemma 4.4, in $\Omega_{Y_0,2R}$ only finitely many zeros occur for each t, and the total number of zero branches intersecting $\Omega_{Y_0,2R}$ over I is finite. Collision-freeness implies that for each $t \in I$ the minimum pairwise separation of zero branches in $\Omega_{Y_0,2R}$ is positive. By compactness of I and finiteness of branches, there exists $\delta > 0$ such that

$$|z_j(t) - z_k(t)| \ge 2\delta \qquad (t \in I,\ j \ne k,\ z_j, z_k \in \Omega_{Y_0,2R}).$$

For each such $z_j(t)$ consider the disc $D_j(t) = \{z : |z - z_j(t)| \le \delta\}$; no other zero lies in $D_j(t)$. By analyticity, $E_z(t, \cdot)$ has no zeros on $\partial D_j(t)$ for δ sufficiently small (uniformly over finitely many branches). By lemma 4.3 and Cauchy estimates on $\partial D_j(t)$,

$$\sup_{z \in \partial D_j(t)} |E_{zz}(t, z)| \le C_1, \qquad \inf_{z \in \partial D_j(t)} |E_z(t, z)| \ge C_2 > 0,$$

with constants independent of j, t (since $|z| \le 3R$ on all such discs and $t \in I$). The maximum principle applied to E_{zz}/E_z shows

$$\Big| \frac{E_{zz}}{E_z}(t, z_j(t)) \Big| \le \frac{C_1}{C_2} =: \widetilde{L}_I.$$

Since $\dot{z}_j(t) = E_{zz}/E_z(t, z_j(t))$ by lemma 4.6, we may take $L_I = \widetilde{L}_I$. □

Lemma 4.8 (Base-slice positivity at a real-zero time)**.** *Let E be an even real entire function of Cartwright class whose zeros (counted with multiplicity) all lie on $\mathbb{R}$. Then*

$$\Im\big(- E'(z)/E(z) \big) \ > \ 0 \qquad (z \in \mathbb{C}_+).$$

Proof. By the canonical product for Cartwright functions with only real zeros (Levin [5], Ch. II), we can write

$$\frac{E'(z)}{E(z)} = \sum_{\rho \in \mathbb{R}} \frac{m_\rho}{z - \rho} + c,$$

with multiplicities $m_\rho \in \mathbb{N}$, some real constant c, and a locally absolutely convergent sum. For $z = x + iy$ with $y > 0$,

$$\frac{1}{z - \rho} = \frac{x - \rho - iy}{(x - \rho)^2 + y^2}, \qquad \Im \frac{1}{z - \rho} = -\frac{y}{(x - \rho)^2 + y^2} < 0.$$

Each term $\frac{m_\rho}{z-\rho}$ contributes m_ρ copies of this negative imaginary part, while c is real and does not affect the imaginary part. Thus

$$\Im\Big(\frac{E'(z)}{E(z)} \Big) = \sum_{\rho \in \mathbb{R}} m_\rho \, \Im \frac{1}{z - \rho} < 0$$

for all $z \in \mathbb{C}_+$. Equivalently,

$$\Im\big(- E'(z)/E(z) \big) > 0 \qquad (z \in \mathbb{C}_+),$$

as claimed. □

Applying this lemma to E_{t_+} (with all zeros real by de Bruijn) provides the initial positivity $h(\cdot, \cdot, t_+) \ge 0$ used by the backward barrier.

5 Stage 2: Time–local algebraic tubes and derivative bounds

Fix a collision-free window $I = [\underline{t}, \overline{t}] \Subset (0, \infty)$ and a spatial box

$$\Omega_{Y_0,2R} = \{(x, y) : |x| \le 2R,\ y \in [\rho, Y_0]\},$$

with $\rho > 0$ small.

Lemma 5.1 (Slab bounds away from the zero set)**.** *Let $I \Subset (0, \infty)$ be a compact time interval and fix $Y_0 > 0$, $R > 0$. Let*

$$\Omega_{Y_0,2R} \;=\; \{z = x + iy :\ |x| \le 2R,\ 0 < y \le Y_0\},$$

and for $t \in I$ let Z_t denote the (multi)set of zeros of E_t.

Then for every $\rho \in (0, 1]$ there exists a constant $C = C(I, Y_0, R)$ such that for all $t \in I$ and all $z \in \Omega_{Y_0,2R}$ satisfying

$$(z, Z_t) \;\ge\; \rho$$

one has

$$|p(t, z)| \;\le\; C\Big(1 + \rho^{-1}\Big), \qquad |p_x(t, z)| \;\le\; C\Big(1 + \rho^{-2}\Big), \tag{5.1}$$

where $p(t, z) = -\Re\big(E_x(t, z)/E_t(z)\big)$ and $p_x = \partial_x p$. The same bounds hold for h, h_x.

Proof. Fix $I \Subset (0, \infty)$ and $\rho \in (0, 1]$. Throughout the proof all implicit constants may depend on I, Y_0, R and the Cartwright data for Ξ, but not on ρ.

Recall

$$g(t, z) := -\frac{E_t'(z)}{E_t(z)} = p(t, z) + i\, h(t, z).$$

Step 1: Log-derivative as a paired sum and tail convergence. For each fixed $t \in I$, evenness of E_t and the Cartwright theory give a canonical product of the form (cf. Lemma 4.6)

$$E_t(z) = e^{\gamma_t} \prod_{k \ge 1} \Big(1 - \frac{z^2}{\rho_k(t)^2}\Big),$$

where the zeros of E_t are $\{\pm\rho_k(t)\}_{k \ge 1}$ (together with a possible zero at 0, which we suppress for notational simplicity).

Differentiating $\log E_t$,

$$\frac{E_t'(z)}{E_t(z)} = \sum_{k \ge 1} \Big(\frac{1}{z - \rho_k(t)} + \frac{1}{z + \rho_k(t)}\Big) =: \sum_{k \ge 1} G_k(t, z),$$

where

$$G_k(t, z) = \frac{1}{z - \rho_k(t)} + \frac{1}{z + \rho_k(t)} = \frac{2z}{z^2 - \rho_k(t)^2}.$$

Fix $M := 4R + 4Y_0$. On $\Omega_{Y_0,2R}$ we have $|z| \le M$. For all zeros with $|\rho_k(t)| > 2M$ we have $|z| \le M \le \frac{1}{2}|\rho_k(t)|$, so

$$|G_k(t, z)| = \frac{2|z|}{|z^2 - \rho_k(t)^2|} \le \frac{4|z|}{|\rho_k(t)|^2} \le \frac{4M}{|\rho_k(t)|^2}.$$

By the rectangle zero-counting bound (U2), for any $T \ge 1$,

$$N_t(T) := \#\{\rho_k(t) : |\rho_k(t)| \le T\} \;\lesssim\; T \log(2 + T),$$

with an implicit constant independent of $t \in I$. Split the zeros into dyadic shells

$$S_m(t) := \{\rho_k(t) : 2^m < |\rho_k(t)| \le 2^{m+1}\}, \qquad m \ge m_0,$$

where $2^{m_0} > 2M$. Then

$$\sum_{\rho_k \in S_m(t)} |G_k(t,z)| \lesssim \frac{\# S_m(t)}{2^{2m}} \lesssim \frac{2^m \log(2+2^m)}{2^{2m}} = \frac{\log(2+2^m)}{2^m}.$$

The series $\sum_{m \ge m_0} \log(2+2^m)/2^m$ converges, so the tail $\sum_{|\rho_k|>2M} G_k(t,z)$ converges absolutely and uniformly for $|z| \le M$ and $t \in I$. In particular, there exists a constant $C_1 = C_1(I, Y_0, R)$ such that

$$\Big| \sum_{|\rho_k(t)|>2M} G_k(t,z) \Big| \;\le\; C_1 \qquad (t \in I,\ |z| \le M). \tag{5.2}$$

Step 2: Near zeros and the distance condition. Fix $t \in I$ and $z = x+iy \in \Omega_{Y_0,2R}$ with $(z, Z_t) \ge \rho$. Let

$$\mathcal{N}(t) := \{k : |\rho_k(t)| \le 2M\}, \qquad \mathcal{F}(t) := \{k : |\rho_k(t)| > 2M\}.$$

By (U2), $\#\mathcal{N}(t) \le N_*$ for some N_* independent of t.

Since Z_t contains all zeros, including the negatives $-\rho_k(t)$, the condition $(z, Z_t) \ge \rho$ implies

$$|z - \rho_k(t)| \;\ge\; \rho, \qquad |z + \rho_k(t)| \;\ge\; \rho \qquad \text{for all } k.$$

Therefore, for $k \in \mathcal{N}(t)$,

$$|G_k(t,z)| = \Big| \frac{1}{z-\rho_k(t)} + \frac{1}{z+\rho_k(t)} \Big| \le \frac{1}{\rho} + \frac{1}{\rho} = \frac{2}{\rho},$$

and hence

$$\Big| \sum_{k \in \mathcal{N}(t)} G_k(t,z) \Big| \;\le\; \frac{2N_*}{\rho} \;\lesssim\; \rho^{-1}. \tag{5.3}$$

Combining (5.2) and (5.3), we obtain

$$|g(t,z)| = \Big| - \sum_{k \in \mathcal{N}(t)} G_k(t,z) - \sum_{k \in \mathcal{F}(t)} G_k(t,z) \Big| \le C_1 + C_2 \rho^{-1} \le C(1+\rho^{-1}).$$

Since $|p|, |h| \le |g|$, this gives the first inequality in (5.1) for p and h.

Step 3: Bounds for p_x, h_x. Because $g(\cdot,\cdot,t)$ is holomorphic in z, we have $g_x = g_z$, and differentiating the expansion gives

$$g_x(t,z) = -\sum_{k \ge 1} \Big(\frac{1}{(z-\rho_k(t))^2} + \frac{1}{(z+\rho_k(t))^2} \Big) =: \; -\sum_{k \ge 1} G_k'(t,z).$$

For $k \in \mathcal{N}(t)$, the distance condition yields

$$|z - \rho_k(t)| \ge \rho, \qquad |z + \rho_k(t)| \ge \rho,$$

and hence

$$|G_k'(t,z)| \le \frac{1}{\rho^2} + \frac{1}{\rho^2} = \frac{2}{\rho^2},$$

so

$$\Big| \sum_{k \in \mathcal{N}(t)} G_k'(t,z) \Big| \;\le\; \frac{2N_*}{\rho^2} \;\lesssim\; \rho^{-2}. \tag{5.4}$$

For $k \in \mathcal{F}(t)$, we still have $|z| \le M$ and $|\rho_k(t)| > 2M$, so $|z \pm \rho_k(t)| \ge \frac{1}{2}|\rho_k(t)|$ and

$$|G_k'(t,z)| = \Big| -\frac{1}{(z-\rho_k(t))^2} - \frac{1}{(z+\rho_k(t))^2} \Big| \le \frac{4}{|\rho_k(t)|^2} + \frac{4}{|\rho_k(t)|^2} \lesssim \frac{1}{|\rho_k(t)|^2}.$$

Exactly as in Step 1, the series $\sum_{k\in\mathcal{F}(t)} |G_k'(t,z)|$ converges absolutely and uniformly on $\{|z| \le M\}$ for $t \in I$, and is bounded by some $C_3(I, Y_0, R)$.

Therefore

$$|g_x(t,z)| \le C_3 + C_4\rho^{-2} \le C(1+\rho^{-2})$$

for all $t \in I$ and all $z \in \Omega_{Y_0,2R}$ with $(z, Z_t) \ge \rho$. Since $|p_x|, |h_x| \le |g_x|$, this gives the second inequality in (5.1) for p_x and h_x. □

5.1 Smoothed distance and tube

Let $Z_I(t)$ be the set of zeros of E_t in $\Omega_{Y_0,2R}$ and define the Euclidean distance $d_I(t,z) = \mathrm{dist}(z, Z_I(t))$ (with $d_I(t,z) = \infty$ if $Z_I(t)$ is empty, but in that case there is no tube to construct). Fix $\eta \in C_c^\infty(B(0,1))$ with $\int \eta = 1$. For $\varepsilon > 0$ set

$$\eta_\varepsilon(w) := \varepsilon^{-2}\eta(w/\varepsilon), \qquad \delta_\varepsilon(t,\cdot) := d_I(t,\cdot) * \eta_\varepsilon.$$

Then $\delta_\varepsilon \in C^\infty$ in z, and:

Lemma 5.2 (Smoothed distance bounds)**.** *For each collision-free I there exists $C_I < \infty$ such that*

$$\|\nabla\delta_\varepsilon\|_{L^\infty(\Omega_{Y_0,2R})} \le 1, \qquad \|\nabla^2\delta_\varepsilon\|_{L^\infty(\Omega_{Y_0,2R})} \le C_I\,\varepsilon^{-1}, \qquad |\partial_t\delta_\varepsilon| \le L_I.$$

Proof. For each fixed t, the function $d_I(t,\cdot)$ is the distance to a closed set and hence is 1–Lipschitz:

$$|d_I(t,z) - d_I(t,z')| \le |z - z'| \qquad (z, z' \in \mathbb{C}).$$

Convolution with the probability kernel η_ε preserves the Lipschitz constant, so $\|\nabla\delta_\varepsilon\|_\infty \le 1$.

For second derivatives, note that $d_I(t,\cdot)$ is Lipschitz, so its gradient ∇d_I exists a.e. and satisfies $\|\nabla d_I(t,\cdot)\|_\infty \le 1$. In the sense of distributions,

$$\nabla\delta_\varepsilon(t,\cdot) = (\nabla d_I(t,\cdot)) * \eta_\varepsilon,$$

and differentiating once more gives

$$\nabla^2\delta_\varepsilon(t,\cdot) = (\nabla d_I(t,\cdot)) * \nabla\eta_\varepsilon.$$

Hence

$$\|\nabla^2\delta_\varepsilon(t,\cdot)\|_\infty \le \|\nabla d_I(t,\cdot)\|_\infty\,\|\nabla\eta_\varepsilon\|_{L^1} \le \|\nabla\eta_\varepsilon\|_{L^1}.$$

Since $\nabla\eta_\varepsilon(w) = \varepsilon^{-3}(\nabla\eta)(w/\varepsilon)$,

$$\|\nabla\eta_\varepsilon\|_{L^1} = \varepsilon^{-3}\int_{\mathbb{R}^2} |\nabla\eta(w/\varepsilon)|\,dw = \varepsilon^{-3}\varepsilon^2\int_{\mathbb{R}^2} |\nabla\eta(u)|\,du \lesssim \varepsilon^{-1},$$

with an implicit constant depending only on η. This yields the second-derivative bound.

For ∂_t, Lemma 4.7 gives a uniform speed bound $|\dot\rho_k(t)| \le L_I$ for all zero branches in $\Omega_{Y_0,2R}$ and $t \in I$. Thus $d_I(t,z)$ is L_I–Lipschitz in t: for any $t, s \in I$,

$$|d_I(t,z) - d_I(s,z)| \le L_I|t-s|.$$

Convolution in z does not increase the Lipschitz constant in t, so $|\partial_t\delta_\varepsilon(t,z)| \le L_I$ for a.e. t, and the bound extends to all t by continuity in t. □

Define the algebraic tube (soft, strictly positive)

$$\Theta_{\rho,\varepsilon,\gamma}(t,z) = \vartheta\Big(\frac{\delta_\varepsilon(t,z)}{\rho}\Big), \qquad \vartheta(s) = \begin{cases} s^\gamma, & 0 \le s \le 1, \\ \text{smooth, increasing}, & 1 < s < 2, \\ 1, & s \ge 2, \end{cases}$$

with $\gamma > 2$ and uniformized $\varepsilon = \theta\rho$ for a fixed $\theta \in (0,1)$.

Lemma 5.3 (Relative-derivative bounds for the tube)**.** *Let $r = \delta_\varepsilon(t,z)$. On $\Omega_{Y_0,2R} \times I$ one has*

$$\frac{|\partial_x \Theta|}{\Theta} \lesssim \frac{\gamma}{r}, \qquad \frac{|\partial_x^2 \Theta|}{\Theta} \lesssim \frac{\gamma(1+\gamma)}{r^2} + \frac{\gamma}{\varepsilon r} + \frac{1}{\rho^2}, \qquad \frac{|\partial_t \Theta|}{\Theta} \lesssim \frac{\gamma L_I}{r},$$

with constants depending only on γ, θ and η.

Proof. Write $\Theta(t,z) = \vartheta(r/\rho)$ with $r = \delta_\varepsilon(t,z)$. By the chain rule,

$$\partial_x \Theta = \vartheta'(r/\rho)\frac{\partial_x r}{\rho}, \qquad \partial_x^2 \Theta = \vartheta''(r/\rho)\frac{(\partial_x r)^2}{\rho^2} + \vartheta'(r/\rho)\frac{\partial_x^2 r}{\rho},$$

and similarly $\partial_t \Theta = \vartheta'(r/\rho)\partial_t r/\rho$.

For $0 < s \le 1$, $\vartheta(s) = s^\gamma$, so

$$\frac{\vartheta'(s)}{\vartheta(s)} = \frac{\gamma}{s}, \qquad \frac{\vartheta''(s)}{\vartheta(s)} \lesssim \frac{\gamma(1+\gamma)}{s^2}.$$

For $1 < s < 2$, ϑ and its derivatives are smooth and bounded away from 0, so ϑ'/ϑ and ϑ''/ϑ are bounded by a constant (depending only on γ and the choice of interpolation), which is also $\lesssim \gamma/s + \gamma(1+\gamma)/s^2$ after adjusting constants. For $s \ge 2$, Θ is constant and all derivatives vanish.

Using $\|\nabla r\|_\infty \le 1$ and $\|\nabla^2 r\|_\infty \lesssim \varepsilon^{-1}$ from Lemma 5.2, and $|\partial_t r| \le L_I$, we obtain on $\{r < 2\rho\}$:

$$\frac{|\partial_x \Theta|}{\Theta} \lesssim \frac{\gamma}{(r/\rho)}\frac{1}{\rho} = \frac{\gamma}{r},$$

and

$$\frac{|\partial_x^2 \Theta|}{\Theta} \lesssim \frac{\gamma(1+\gamma)}{(r/\rho)^2}\frac{1}{\rho^2} + \frac{\gamma}{(r/\rho)}\frac{1}{\varepsilon\rho} = \frac{\gamma(1+\gamma)}{r^2} + \frac{\gamma}{\varepsilon r}.$$

In the transition region $r \in [\rho, 2\rho]$ the same computation (with the rough bounds $\vartheta'/\vartheta, \vartheta''/\vartheta \lesssim 1$) yields

$$\frac{|\partial_x^2 \Theta|}{\Theta} \lesssim \frac{1}{\rho^2} + \frac{1}{\varepsilon\rho},$$

which is dominated by the displayed bound after adding the harmless $1/\rho^2$ term. Finally,

$$\frac{|\partial_t \Theta|}{\Theta} \lesssim \frac{\gamma}{(r/\rho)}\frac{L_I}{\rho} = \frac{\gamma L_I}{r}.$$

For $r \ge 2\rho$, Θ is constant and all derivatives vanish, so the inequalities hold trivially. ∎

5.2 Local singular profile on the tube

Lemma 5.4 (Local singular profile of g, p, p_x on the tube)**.** *Fix a compact time interval $I \Subset (0, \infty)$ and work in $\Omega_{Y_0,2R}$. For $t \in I$ and for every zero $\rho_j(t)$ with $\Im\rho_j(t) \in [\rho, Y_0]$, set $r = |z - \rho_j(t)|$. On the tube region $\{r \le 2\rho\}$ one has*

$$|g(z,t)| \;\lesssim\; \frac{1}{r} \;+\; \frac{1}{\rho}, \qquad |g_x(z,t)| \;\lesssim\; \frac{1}{r^2} \;+\; \frac{1}{\rho^2},$$

with constants depending only on I, Y_0, R and the U1/U2 data. In particular,

$$|p(z,t)|,\, |h(z,t)| \;\le\; |g(z,t)| \;\lesssim\; \frac{1}{r} + \frac{1}{\rho}, \qquad |p_x(z,t)|,\, |h_x(z,t)| \;\le\; |g_x(z,t)| \;\lesssim\; \frac{1}{r^2} + \frac{1}{\rho^2}.$$

Proof. From the canonical product (see Lemma 4.6) and evenness, we can write

$$\frac{E_t'(z)}{E_t(z)} = \sum_k \frac{1}{z - \rho_k(t)},$$

with the sum locally absolutely convergent in z. Thus

$$g(z,t) = -\frac{E_t'(z)}{E_t(z)} = -\sum_k \frac{1}{z - \rho_k(t)} = p(z,t) + ih(z,t).$$

Fix a zero $\rho_j(t)$ with $\Im\rho_j(t) \in [\rho, Y_0]$, and split the sum into three parts:

$$g(z,t) = \frac{1}{z - \rho_j(t)} + \sum_{k \ne j,\ |\rho_k - \rho_j| \le 1} \frac{1}{z - \rho_k(t)} + \sum_{|\rho_k - \rho_j| > 1} \frac{1}{z - \rho_k(t)}.$$

On the tube $\{r = |z - \rho_j(t)| \le 2\rho\}$, the principal term contributes $O(r^{-1})$ to g. For the "near" sum with $|\rho_k - \rho_j| \le 1$, note that by definition of r we always have

$$|z - \rho_k(t)| \;\ge\; r \qquad \text{for every zero } \rho_k(t),$$

so

$$\sum_{k \ne j,\ |\rho_k - \rho_j| \le 1} \frac{1}{|z - \rho_k(t)|} \;\le\; \frac{N_{\text{near}}(t)}{r},$$

where $N_{\text{near}}(t)$ is the number of zeros in the disc $B(\rho_j(t), 1)$. By the rectangle zero-counting bound (U2), this number is bounded uniformly in $t \in I$ and j by a constant depending only on I, Y_0, R and the U2 data. Thus the near sum contributes $O(r^{-1})$.

For the tail over $|\rho_k - \rho_j| > 1$, cover the zeros by dyadic annuli

$$A_m = \{\rho_k : 2^m < |\rho_k - \rho_j| \le 2^{m+1}\}, \qquad m = 0, 1, \dots.$$

For z with $r \le 2\rho \le 1$, we have $|z - \rho_k| \ge \frac{1}{2}|\rho_k - \rho_j|$ on A_m for $m \ge 0$, so

$$\sum_{\rho_k \in A_m} \frac{1}{|z - \rho_k|} \lesssim \frac{\#A_m}{2^m}.$$

Cartwright zero-counting (U2) gives $\#A_m \lesssim 2^m \log(2 + 2^m)$, so

$$\sum_{\rho_k \in A_m} \frac{1}{|z - \rho_k|} \lesssim \log(2 + 2^m).$$

Summing over $m \le C\log(2+R)$ (since all zeros under consideration lie in a bounded box) produces a bound depending only on R, uniform in t and j. Hence

$$|g(z,t)| \lesssim \frac{1}{r} + C(R) \lesssim \frac{1}{r} + \frac{1}{\rho}$$

on $\{r \le 2\rho\}$ (after enlarging the constant and using $\rho \le 1$), and the same bound holds for $|p(z,t)|, |h(z,t)| \le |g(z,t)|$.

For g_x, differentiate the series in x (which coincides with differentiation in z by holomorphy):

$$g_x(z,t) = -\sum_k \frac{1}{(z-\rho_k(t))^2}.$$

The principal term gives $O(r^{-2})$. For the near sum with $|\rho_k - \rho_j| \le 1$ we again use $|z-\rho_k| \ge r$ to get

$$\sum_{k\ne j,\ |\rho_k-\rho_j|\le 1} \frac{1}{|z-\rho_k|^2} \ \le\ \frac{N_{\mathrm{near}}(t)}{r^2} \ \lesssim\ \frac{1}{r^2},$$

with the same uniform bound on $N_{\mathrm{near}}(t)$ as above. For the tail on A_m,

$$\sum_{\rho_k\in A_m} \frac{1}{|z-\rho_k|^2} \lesssim \frac{\#A_m}{2^{2m}} \lesssim \frac{\log(2+2^m)}{2^m},$$

and the sum over m converges absolutely. Thus

$$|g_x(z,t)| \lesssim \frac{1}{r^2} + C'(R) \lesssim \frac{1}{r^2} + \frac{1}{\rho^2}$$

on $\{r \le 2\rho\}$, and therefore the same bound holds for $|p_x(z,t)|, |h_x(z,t)| \le |g_x(z,t)|$. □

5.3 Tube integral bookkeeping

Let W be the weight used in the barrier (defined in §6) and $w = Wh^-$. On $\Omega_{Y_0,2R}$ we assume W has the form

$$W(t,z) = \chi(y)\, y^\alpha\, \Theta_{\rho,\varepsilon,\gamma}(t,z),$$

with χ bounded, $y \in [\rho, Y_0]$, $\alpha > 0$, and $\Theta_{\rho,\varepsilon,\gamma}$ as above.

We will need to control tube integrals of w with powers of r^{-1}.

Lemma 5.5 (Tube integral bookkeeping)**.** *Let $w = Wh^-$ with W as in §6 and $\gamma > 2$. Then on each tube $\{r \le 2\rho\}$ one has*

$$\int_{r\le 2\rho} w^2 \ \lesssim\ 1,$$

$$\int_{r\le 2\rho} \frac{w^2}{r^2} \ \lesssim\ \rho^{-2},$$

$$\int_{r\le 2\rho} \frac{w^2}{r^4} \ \lesssim\ \rho^{-4},$$

with constants depending on $I, Y_0, R, \alpha, \gamma, \theta$ and the U1/U2 constants.

Proof. On the tube, $y \in [\rho, Y_0]$ and Lemma 5.4 gives

$$|g(z,t)| \lesssim \frac{1}{r} + \frac{1}{\rho},$$

hence

$$|h^-(z,t)| \le |h(z,t)| \le |g(z,t)| \lesssim \frac{1}{r} + \frac{1}{\rho}.$$

The weight W contains $\chi(y)$, y^α and Θ, with $|\chi(y)| \le C_\chi$ and $y^\alpha \le Y_0^\alpha$ on the strip. Thus

$$|W(t,z)| \lesssim \Theta_{\rho,\varepsilon,\gamma}(t,z),$$

and on $\{r \le \rho\}$ we have $\Theta \approx (r/\rho)^\gamma$, while on $\{\rho \le r \le 2\rho\}$ we have $\Theta \approx 1$ (up to multiplicative constants depending only on γ and the choice of interpolation).

Hence

$$w^2 \lesssim \Theta^2 \Big(\frac{1}{r} + \frac{1}{\rho}\Big)^2 \lesssim \Theta^2 \Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big).$$

Estimate of $\int_{r \le 2\rho} w^2$. Split the tube into $r \le \rho$ and $\rho \le r \le 2\rho$.

On $r \le \rho$, $\Theta \lesssim (r/\rho)^\gamma$, so

$$w^2 \lesssim \Big(\frac{r}{\rho}\Big)^{2\gamma} \Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big).$$

In polar coordinates around $\rho_j(t)$, the area element is $r\,dr\,d\theta$, and

$$\begin{aligned} \int_{r \le \rho} w^2 &\lesssim \int_0^{2\pi} \int_0^\rho \Big(\frac{r}{\rho}\Big)^{2\gamma} \Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big) r\,dr\,d\theta \\ &\lesssim \int_0^\rho \frac{r^{2\gamma-1}}{\rho^{2\gamma}}\,dr + \int_0^\rho \frac{r^{2\gamma+1}}{\rho^{2\gamma+2}}\,dr \ \lesssim\ 1. \end{aligned}$$

On $\rho \le r \le 2\rho$, $\Theta \approx 1$, so

$$w^2 \lesssim \frac{1}{r^2} + \frac{1}{\rho^2},$$

and

$$\int_{\rho \le r \le 2\rho} w^2 \lesssim \int_\rho^{2\rho} \Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big) r\,dr \lesssim 1.$$

Combining the two regions yields $\int_{r \le 2\rho} w^2 \lesssim 1$.

Estimate of $\int_{r \le 2\rho} w^2/r^2$. On $r \le \rho$,

$$\frac{w^2}{r^2} \lesssim \Big(\frac{r}{\rho}\Big)^{2\gamma} \Big(\frac{1}{r^4} + \frac{1}{\rho^2 r^2}\Big),$$

so

$$\begin{aligned} \int_{r \le \rho} \frac{w^2}{r^2} &\lesssim \int_0^\rho \Big(\frac{r}{\rho}\Big)^{2\gamma} \Big(\frac{1}{r^4} + \frac{1}{\rho^2 r^2}\Big) r\,dr \\ &\lesssim \frac{1}{\rho^{2\gamma}} \int_0^\rho r^{2\gamma-3}\,dr + \frac{1}{\rho^{2\gamma+2}} \int_0^\rho r^{2\gamma-1}\,dr \ \lesssim\ \rho^{-2}, \end{aligned}$$

using $\gamma > 1$. On $\rho \le r \le 2\rho$,

$$\frac{w^2}{r^2} \lesssim \frac{1}{r^4} + \frac{1}{\rho^2 r^2},$$

and
$$\int_{\rho\le r\le 2\rho}\frac{w^2}{r^2}\lesssim\int_\rho^{2\rho}\Big(\frac{1}{r^4}+\frac{1}{\rho^2r^2}\Big)r\,dr\lesssim\frac{1}{\rho^2}.$$
Thus $\int_{r\le 2\rho}w^2/r^2\lesssim\rho^{-2}$.

Estimate of $\displaystyle\int_{r\le 2\rho}w^2/r^4$. On $r\le\rho$,
$$\frac{w^2}{r^4}\lesssim\Big(\frac{r}{\rho}\Big)^{2\gamma}\Big(\frac{1}{r^6}+\frac{1}{\rho^2r^4}\Big),$$
so
$$\begin{aligned}\int_{r\le\rho}\frac{w^2}{r^4}&\lesssim\int_0^\rho\Big(\frac{r}{\rho}\Big)^{2\gamma}\Big(\frac{1}{r^6}+\frac{1}{\rho^2r^4}\Big)r\,dr\\&\lesssim\frac{1}{\rho^{2\gamma}}\int_0^\rho r^{2\gamma-5}\,dr+\frac{1}{\rho^{2\gamma+2}}\int_0^\rho r^{2\gamma-3}\,dr\ \lesssim\ \rho^{-4},\end{aligned}$$
using $\gamma>2$. On $\rho\le r\le 2\rho$,
$$\frac{w^2}{r^4}\lesssim\frac{1}{r^6}+\frac{1}{\rho^2r^4},$$
and
$$\int_{\rho\le r\le 2\rho}\frac{w^2}{r^4}\lesssim\int_\rho^{2\rho}\Big(\frac{1}{r^6}+\frac{1}{\rho^2r^4}\Big)r\,dr\lesssim\rho^{-4}.$$
This proves the claimed bounds. □

Remark 5.6. The exponents of r in the above integrals reduce to the integrability of $r^{2\gamma-3}$ and $r^{2\gamma-5}$ at 0, so $\gamma>1$ suffices for $\int w^2/r^2$ and $\gamma>2$ for $\int w^2/r^4$. We keep $\gamma>2$ to comfortably cover second-derivative terms arising from W_{xx} and $(h^-)_x$ in the barrier argument.

6 Stage 3: Backward Carleman–Kato barrier on a short window

In this section we prove the short-time backward barrier that underlies Stage 3. The key point is that we use only the *weak* Kato inequality with divergence-form drift (Lemma B.1), in which the negative part h^- is a distributional supersolution up to a nonnegative defect measure. Together with a bilinear Carleman identity for

$$L := \partial_t + \partial_x^2$$

this suffices to obtain a robust barrier estimate on short time windows, *provided we start from a time at which the negative part has vanished.* In the global argument this input comes from the real-zero slice at t_+ (Lemma 4.8) and from the collision-bridging mechanism of Stage 4.

Throughout we fix a collision-free time window

$$I = [t_1 - \Delta, t_1] \subset (0, \infty)$$

contained in one of the components produced in Stage 2, and work in a spatial box $\{|x| \le 2R,\ y \in (0, Y_0]\}$. All implicit constants may depend on the fixed background data (Cartwright parameters, Y_0, γ, etc.) but are independent of the large Carleman parameter λ and the small tube radius ρ and window length Δ.

We will only apply the barrier on windows for which

$$h^-(t_1, x, y) \equiv 0 \qquad \text{for all } (x, y) \in \mathbb{C}_+, \tag{6.1}$$

i.e. the logarithmic derivative has nonnegative imaginary part at the top time t_1. In the global scheme, such times arise either as the base slice t_+ from de Bruijn or as restart times below collisions (Stage 4).

6.1 Window, weights, and the barrier unknown

On the window $I = [t_1 - \Delta, t_1]$ choose a smooth time cutoff $\eta \in C_c^\infty((t_1 - \Delta - \tau, t_1 + \tau))$ with $0 \le \eta \le 1$,

$$\eta(t) \equiv 1 \text{ for } t \in [t_1 - \Delta + \tau, t_1], \qquad \eta(t_1 - \Delta) = \eta(t_1 + \tau) = 0, \qquad |\eta'(t)| \lesssim \tau^{-1}, \tag{6.2}$$

for some $\tau \in (0, \Delta/4)$. Thus η vanishes at the lower endpoint of I and equals 1 all the way up to the top time t_1.

As in Stage 2 we fix a tube radius $\rho > 0$, a smoothing parameter $\varepsilon = \theta\rho$ with $\theta \in (0, 1)$, and an exponent $\gamma > 2$, and form the time–local algebraic tube $\Theta_{\rho,\varepsilon,\gamma}(t, z)$ around $Z_I(t)$.

We also choose vertical and horizontal cutoffs

$$\chi \in C_c^\infty((0, 2Y_0)), \quad 0 \le \chi \le 1, \quad \chi(y) \equiv 1 \text{ on } [c\rho, Y_0],$$

$$\omega_R \in C_c^\infty(\mathbb{R}), \quad 0 \le \omega_R \le 1, \quad \omega_R(x) \equiv 1 \text{ on } [-R, R], \quad \operatorname{supp} \omega_R \subset [-2R, 2R],$$

for some fixed $c \in (0, 1)$. Fix $\alpha > 2$ and define

$$W(t, z) = \eta(t)\, \chi(y)\, y^\alpha\, \Theta_{\rho,\varepsilon,\gamma}(t, z)\, \omega_R(x), \qquad z = x + iy \in \mathbb{C}_+. \tag{6.3}$$

By construction $W \ge 0$, $W > 0$ on the interior of the spacetime region where the barrier will be applied, and

$$W(t_1 - \Delta, \cdot) \equiv 0, \qquad W(t_1, \cdot) > 0 \text{ on } \{|x| \le R,\ c\rho \le y \le Y_0\}.$$

On zero-free regions, Lemma 4.1 gives the drifted backward equation

$$(\partial_t + \partial_x^2)h = \partial_x(2ph), \tag{6.4}$$

where

$$p(t,x,y) = -\Re\frac{\partial_x E_t(x+iy)}{E_t(x+iy)}, \qquad h(t,x,y) = \Im\Big(-\frac{\partial_x E_t}{E_t}(x+iy)\Big).$$

We denote by $h^-(t,x,y) := \max\{-h(t,x,y), 0\}$ the negative part, and define the barrier unknown

$$w(t,x,y) := W(t,x,y)\, h^-(t,x,y).$$

For each fixed $y > 0$, we will apply a one-dimensional Carleman identity in x to $w(\cdot,\cdot,y)$ and then integrate over y. Note that by (6.1) and the definition of W we have

$$w(t_1,x,y) = 0, \qquad w(t_1-\Delta,x,y) = 0 \quad \text{for all } (x,y). \tag{6.5}$$

6.2 Backward Carleman identity in bilinear form

We first record a bilinear identity for the backward heat operator. Set

$$L := \partial_t + \partial_x^2, \qquad \phi(t) := \frac{\lambda}{t_1 - t}, \qquad t < t_1,\ \lambda > 0.$$

Lemma 6.1 (Backward Carleman identity)**.** *Let $w = w(t,x)$ be smooth and compactly supported in x, supported in $[t_1-\Delta, t_1]$ in t, and satisfy*

$$w(t_1-\Delta,\cdot) = w(t_1,\cdot) = 0.$$

Define

$$B(w) := \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t)\big(|\partial_x w|^2 + \phi'(t)w^2\big)\, e^{2\phi(t)}\, dx\, dt,$$

$$N(w) := \frac{1}{2}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} w^2 e^{2\phi(t)}\, dx\, dt.$$

Then

$$B(w) = -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t)\, Lw\, w\, e^{2\phi(t)}\, dx\, dt + N(w). \tag{6.6}$$

Proof. Set $v := we^{\phi}$. Then $w = e^{-\phi}v$ and

$$Lw = w_t + w_{xx} = e^{-\phi}(v_t - \phi' v + v_{xx}),$$

hence

$$Lw\, w\, e^{2\phi} = (v_t - \phi' v + v_{xx})v.$$

Define

$$J(w) := -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t) Lw\, w\, e^{2\phi}\, dx\, dt = -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t)(v_t - \phi' v + v_{xx})v\, dx\, dt.$$

Integrating by parts in t and x and using the vanishing of w (hence v) at $t = t_1 - \Delta$ and $t = t_1$, we obtain

$$\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t) v_t v\, dx\, dt = \frac{1}{2}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} v^2\, dx\, dt,$$

$$\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t) v_{xx} v\, dx\, dt = -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t)|v_x|^2\, dx\, dt,$$

and

$$\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)(-\phi' v)v\,dx\,dt = -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)\phi' v^2\,dx\,dt.$$

Thus

$$J(w) = -\frac{1}{2}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} v^2\,dx\,dt + \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)\big(|v_x|^2+\phi' v^2\big)\,dx\,dt,$$

which, after returning to w and factoring out $e^{2\phi}$, is exactly (6.6). $\square$

Remark 6.2 (Applicability to $w = Wh^-$). For each fixed $y > 0$, the function $w = Wh^-$ constructed above is compactly supported in x, supported in $[t_1-\Delta, t_1]$ in t, and satisfies (6.5), so the hypotheses of Lemma 6.1 are met.

6.3 Weak Kato decomposition for h^-

On the window I and away from the zeros of E_t, the drifted backward equation (6.4) holds pointwise. Lemma B.1 applied on such a zero-free region (for each fixed $y > 0$) yields a *weak Kato decomposition*

$$(\partial_t + \partial_x^2)h^- - \partial_x(2ph^-) = \mu, \tag{6.7}$$

where $\mu \geq 0$ is a nonnegative Radon measure on $I \times \mathbb{R}_x$ (depending on y as a parameter). Equivalently,

$$Lh^- = \partial_x(2ph^-) + \mu, \qquad \mu \geq 0.$$

We now plug this into the Carleman identity (6.6) with $w = Wh^-$.

6.4 Decomposition of Lw and splitting of the Carleman functional

Fix $y > 0$ and suppress it from the notation. Using $w = Wh^-$ and $L = \partial_t + \partial_x^2$,

$$Lw = L(Wh^-) = WLh^- + W_t h^- + 2W_x(h^-)_x + W_{xx}h^-.$$

Using (6.7), this becomes

$$Lw = W\partial_x(2ph^-) + W\mu + R,$$

where

$$R := W_t h^- + 2W_x(h^-)_x + W_{xx}h^-. \tag{6.8}$$

Let

$$J(w) := -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)Lw\,w\,e^{2\phi}\,dx\,dt$$

be as in Lemma 6.1. Substituting the decomposition of Lw yields

$$J(w) = J_{\text{drift}} + J_\mu + J_{\text{comm}},$$

where

$$J_{\text{drift}} := -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)\,W\partial_x(2ph^-)\,w\,e^{2\phi}\,dx\,dt, \tag{6.9}$$

$$J_\mu := -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)\,W\mu\,w\,e^{2\phi}\,dx\,dt, \tag{6.10}$$

$$J_{\text{comm}} := -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1-t)\,R\,w\,e^{2\phi}\,dx\,dt. \tag{6.11}$$

Since $W \ge 0$, $w = Wh^- \ge 0$, $(t_1 - t)e^{2\phi} \ge 0$ and $\mu \ge 0$, we have

$$J_\mu \le 0.$$

Hence, from (6.6),

$$B(w) = J(w) + N(w) \le J_{\mathrm{drift}} + J_{\mathrm{comm}} + N(w) \le |J_{\mathrm{drift}}| + |J_{\mathrm{comm}}| + N(w). \tag{6.12}$$

Crucially, the Kato defect measure contributes only a favourable (nonpositive) term, and can be discarded in the barrier estimate.

6.5 Estimate of the drift term J_{drift}

Using (6.9) and $w = Wh^-$,

$$J_{\mathrm{drift}} = -\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, W\partial_x(2ph^-)Wh^-\, e^{2\phi}\, dx\, dt.$$

Integrating by parts in x (using the compact support of ω_R),

$$\begin{aligned} J_{\mathrm{drift}} &= \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, 2ph^-\, \partial_x\big(W^2h^-e^{2\phi}\big)\, dx\, dt \\ &= \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, 2ph^-\Big[W^2(h^-)_x + 2WW_xh^-\Big]e^{2\phi}\, dx\, dt. \end{aligned}$$

Expressing $(h^-)_x$ in terms of w and w_x,

$$h^- = \frac{w}{W}, \qquad (h^-)_x = \frac{1}{W}w_x - \frac{W_x}{W^2}w = \frac{1}{W}w_x - \frac{W_x}{W}h^-,$$

we have $W^2(h^-)_x = Ww_x - WW_xh^-$, so

$$J_{\mathrm{drift}} = \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, 2p\Big(ww_x + \frac{W_x}{W}w^2\Big)e^{2\phi}\, dx\, dt,$$

and hence

$$|J_{\mathrm{drift}}| \lesssim \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, |p|\Big(|w_x||w| + \Big|\frac{W_x}{W}\Big|w^2\Big)e^{2\phi}\, dx\, dt. \tag{6.13}$$

We split the integral into the plateau and tube regions defined from the tube weight Θ:

$$\mathcal{A}_\tau := \{\Theta_{\rho,\varepsilon,\gamma} \ge \tau\}, \qquad \mathcal{B}_\tau := \{\Theta_{\rho,\varepsilon,\gamma} < \tau\},$$

for some fixed $\tau \in (0, 1)$.

Plateau region $\mathcal{A}_\tau$. On $\mathcal{A}_\tau$ the smoothed distance $r = \delta_\varepsilon(t, z)$ to the zero set satisfies $r \gtrsim_\tau \rho$ (since for $r \le \rho$ the relation $\Theta = (r/\rho)^\gamma \ge \tau$ forces $r \ge \tau^{1/\gamma}\rho$). For ρ small and $\varepsilon = \theta\rho$ with fixed $\theta \in (0, 1)$, this implies the true distance to $Z_I(t)$ is also $\gtrsim_\tau \rho$. Applying Lemma 5.1 with this lower bound and using $\rho \le 1$ yields

$$|p(t, z)| \;\lesssim\; 1 + \rho^{-1} \;\lesssim\; \rho^{-1} \qquad \text{on } \mathcal{A}_\tau.$$

Moreover, by Lemma 5.3 and the structure of W,

$$\Big|\frac{W_x}{W}\Big| \lesssim 1 + \frac{1}{r} + \frac{1}{\rho} \lesssim 1 + \rho^{-1} \lesssim \rho^{-1} \qquad \text{on } \mathcal{A}_\tau.$$

Thus, on $\mathcal{A}_\tau$,

$$|p|\Big(|w_x||w| + \Big|\frac{W_x}{W}\Big|w^2\Big) \lesssim \rho^{-1}(|w_x||w| + \rho^{-1}w^2).$$

By Cauchy–Schwarz and Young,

$$\int_{\mathcal{A}_\tau} (t_1 - t)\rho^{-1}|w_x||w|e^{2\phi} \le \varepsilon B(w) + C(\varepsilon)\rho^{-2}\int_{\mathcal{A}_\tau}(t_1 - t)w^2 e^{2\phi},$$

and the w^2 term is bounded similarly. Altogether,

$$|J^{\mathcal{A}}_{\mathrm{drift}}| \lesssim \varepsilon B(w) + C(\varepsilon)\rho^{-2}\int_{\mathcal{A}_\tau}(t_1 - t)w^2 e^{2\phi}\,dx\,dt.$$

Tube region $\mathcal{B}_\tau$. On $\mathcal{B}_\tau$ we are very close to the zero set. Lemmas 5.4 and 5.3 give, with r the (smoothed) distance to the nearest zero,

$$|p| \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad \Big|\frac{W_x}{W}\Big| \lesssim \frac{1}{r} + \frac{1}{\rho}.$$

Hence

$$|J^{\mathcal{B}}_{\mathrm{drift}}| \lesssim \int_{\mathcal{B}_\tau}(t_1 - t)\Big(\Big(\frac{1}{r} + \frac{1}{\rho}\Big)|w_x||w| + \Big(\frac{1}{r} + \frac{1}{\rho}\Big)w^2\Big)e^{2\phi}\,dx\,dt.$$

Applying Cauchy–Schwarz and Young again yields terms of the form

$$\int_{\mathcal{B}_\tau}(t_1 - t)\Big(|w_x|^2 + \Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big)w^2\Big)e^{2\phi}\,dx\,dt.$$

The $|w_x|^2$ term is absorbed into $B(w)$, while the $r^{-2}w^2$ terms are converted into $\rho^{-2}w^2$ using the tube integral bounds in Lemma 5.5. In particular, the tube geometry produces an additional ρ^{-2}, so that

$$|J^{\mathcal{B}}_{\mathrm{drift}}| \le \varepsilon B(w) + C(\varepsilon)\int_{\mathcal{B}_\tau}(t_1 - t)(\rho^{-2} + \rho^{-4})w^2 e^{2\phi}\,dx\,dt.$$

Conclusion for J_{drift}. Combining plateau and tube contributions, and noting that the ρ^{-2} term from $\mathcal{A}_\tau$ is dominated by the $(\rho^{-2} + \rho^{-4})$ factor, we arrive at

$$|J_{\mathrm{drift}}| \le \varepsilon B(w) + C(\varepsilon)\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1 - t)(\rho^{-2} + \rho^{-4})w^2 e^{2\phi}\,dx\,dt. \tag{6.14}$$

6.6 Estimate of the commutator term J_{comm}

From (6.8) and $w = Wh^-$,

$$|R| \lesssim \Big(\Big|\frac{W_t}{W}\Big| + \Big|\frac{W_{xx}}{W}\Big|\Big)w + \Big|\frac{W_x}{W}\Big||w_x|,$$

so

$$|J_{\mathrm{comm}}| \lesssim \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}}(t_1 - t)\Big(\Big(\Big|\frac{W_t}{W}\Big| + \Big|\frac{W_{xx}}{W}\Big|\Big)w^2 + \Big|\frac{W_x}{W}\Big||w_x||w|\Big)e^{2\phi}\,dx\,dt.$$

By Lemma 5.3 and the structure of $\eta, \chi, y^\alpha, \omega_R$, we have on the truncated spacetime region Ω_I:

$$\Big|\frac{W_x}{W}\Big| \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad \Big|\frac{W_{xx}}{W}\Big| \lesssim \frac{1}{r^2} + \frac{1}{\rho^2} + \Delta^{-2},$$

$$\Big|\frac{W_t}{W}\Big| \lesssim \frac{L_I}{r} + \Delta^{-1},$$

where L_I is the local speed bound from Lemma 4.7, and the Δ^{-1}, Δ^{-2} contributions come from the lower time shoulder (the upper endpoint contributes no extra terms thanks to (6.1)). Splitting into plateau and tube, using Cauchy–Schwarz/Young, and applying Lemma 5.5 to handle r^{-2}, r^{-4}, we obtain

$$|J_{\mathrm{comm}}| \le \varepsilon B(w) + C(\varepsilon) \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)(\rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2}) w^2 e^{2\phi} \, dx \, dt. \tag{6.15}$$

6.7 Control of the zeroth-order term and cubic coupling

From the definition of $B(w)$ and $\phi'(t) = \lambda/(t_1 - t)^2$, we have

$$\int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)\phi'(t) w^2 e^{2\phi} \, dx \, dt = \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} \frac{\lambda}{t_1 - t} w^2 e^{2\phi} \, dx \, dt \ge \frac{\lambda}{\Delta} \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} w^2 e^{2\phi} \, dx \, dt,$$

so

$$N(w) = \frac{1}{2} \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} w^2 e^{2\phi} \, dx \, dt \le \frac{\Delta}{2\lambda} B(w). \tag{6.16}$$

Combining (6.12), (6.14), (6.15) and (6.16), for any fixed $\varepsilon > 0$ and $\lambda > 0$ we obtain

$$\begin{aligned} B(w) &\le |J_{\mathrm{drift}}| + |J_{\mathrm{comm}}| + N(w) \\ &\le 2\varepsilon B(w) + C(\varepsilon) \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)(\rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2}) w^2 e^{2\phi} \, dx \, dt \\ &\quad + \frac{\Delta}{2\lambda} B(w). \end{aligned}$$

Choose $\varepsilon > 0$ small and then λ large (relative to Δ) so that

$$2\varepsilon + \frac{\Delta}{2\lambda} \le \frac{1}{2}.$$

Absorbing these contributions into the left-hand side yields

$$B(w) \lesssim \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)(\rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2}) w^2 e^{2\phi} \, dx \, dt. \tag{6.17}$$

We now impose the *cubic coupling*

$$\rho = \lambda^{-1/2}, \qquad \Delta \ \le \ c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}\},$$

for a sufficiently small absolute constant $c_0 > 0$. A straightforward computation shows that, under this coupling and for λ large, each coefficient

$$\frac{(t_1 - t)\rho^{-2}}{\phi'(t)}, \quad \frac{(t_1 - t)\rho^{-4}}{\phi'(t)}, \quad \frac{(t_1 - t)L_I^2 \rho^{-2}}{\phi'(t)}, \quad \frac{(t_1 - t)\Delta^{-2}}{\phi'(t)}$$

is $\ll 1$ on $[t_1 - \Delta, t_1)$. Therefore the right-hand side of (6.17) is strictly dominated by the ϕ'-term inside $B(w)$, and (6.17) forces $B(w) = 0$.

Since the integrand in $B(w)$ is nonnegative, this implies

$$\partial_x w \equiv 0, \qquad w \equiv 0 \quad \text{a.e. on } [t_1 - \Delta, t_1] \times \mathbb{R}_x$$

for each fixed y in the support of $\chi(y)$. On the interior spacetime region where $\eta, \chi, \Theta, \omega_R$ are strictly positive, $W > 0$, so $w = Wh^- = 0$ implies $h^- \equiv 0$ there, i.e. $h \ge 0$.

Integrating over y and undoing the cutoffs via the exhaustion of Stage 5 yields the short-time barrier:

Proposition 6.3 (Short-time backward barrier)**.** *Let $I = [t_1 - \Delta, t_1]$ be a collision-free time window and assume h solves (6.4) on the corresponding zero-free spacetime region $\Omega_I \subset \mathbb{C}_+$. Suppose moreover that at the top time t_1 one has*

$$h^-(t_1, x, y) \equiv 0 \qquad \text{for all } (x, y) \in \mathbb{C}_+. \tag{6.18}$$

Then there exist $\lambda_0 > 0$ and $c_0 > 0$, depending only on the fixed data, such that for all $\lambda \geq \lambda_0$ and

$$\rho = \lambda^{-1/2}, \qquad \Delta \ \leq \ c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}\},$$

the negative part h^- vanishes on the window, i.e.

$$h(t, x, y) \geq 0 \quad \text{for all } z = x + iy \in \mathbb{C}_+, \ t \in [t_1 - \Delta, t_1].$$

In the global argument, the input (6.18) comes either from the base slice t_+ (via Lemma 4.8) or from the restart times provided by the collision-bridging lemma in Stage 4. Iterating this short-time barrier across a chain of such windows, and then exhausting in (x, y) as in Stage 5, yields global backward positivity of h on $\mathbb{C}_+$ for all $t \leq t_+$, as summarized in Section 3.

Lemma 6.4 (Quantitative Carleman absorption under the cubic coupling)**.** *Let $I = [t_1 - \Delta, t_1]$ be a collision-free window and L_I the local speed bound from Lemma 4.7. Let $w = Wh^-$ be as in §6, and let $B(w)$ be the Carleman bulk*

$$B(w) := \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)\big(|\partial_x w|^2 + \phi'(t) w^2\big) e^{2\phi(t)} \, dx \, dt, \qquad \phi(t) = \frac{\lambda}{t_1 - t}.$$

There exists a constant $C_0 \geq 1$, depending only on the fixed background data (Cartwright parameters, $Y_0, \alpha, \gamma, \theta$ and the choice of cutoffs), such that for all $\lambda > 0$, $\rho > 0$, $\Delta > 0$ for which the estimates of §6 apply, we have

$$B(w) \ \leq \ C_0 \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t) \, A(\lambda, \rho, \Delta, L_I) \, w^2 e^{2\phi} \, dx \, dt, \tag{6.19}$$

where

$$A(\lambda, \rho, \Delta, L_I) := \rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2}.$$

Assume now that

$$\rho = \lambda^{-1/2}, \qquad 0 < \Delta \leq c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}, 1\}, \tag{6.20}$$

for some fixed constant $c_0 > 0$, and that $\lambda \geq \lambda_*$ for a sufficiently large constant λ_* depending only on the same background data (but not on L_I, Δ, ρ, R). Then, for c_0 chosen sufficiently small and λ_* sufficiently large (both depending only on C_0), the inequality (6.19) forces

$$B(w) = 0.$$

In particular, on the corresponding spacetime region one has $w \equiv 0$, hence $h^- = 0$ wherever $W > 0$.

Proof. The first part, (6.19), is just a quantified version of (6.17): the symbol $\lesssim$ there means precisely that there exists $C_0 \geq 1$ depending only on the structural data such that (6.19) holds. We now show that, under the cubic coupling (6.20), the right-hand side of (6.19) is a strict contraction of $B(w)$.

Step 1: Compare the error weight with the Carleman bulk. Set

$$A := A(\lambda, \rho, \Delta, L_I) = \rho^{-2} + \rho^{-4} + L_I^2 \rho^{-2} + \Delta^{-2}.$$

Then (6.19) reads

$$B(w) \;\le\; C_0 \int_{t_1-\Delta}^{t_1}\!\!\int_{\mathbb{R}} (t_1-t)\,A\,w^2 e^{2\phi}\,dx\,dt. \tag{6.21}$$

We want to bound the right-hand side in terms of $B(w)$ itself. Recall that the w^2-part of $B(w)$ is

$$B_2(w) := \int_{t_1-\Delta}^{t_1}\!\!\int_{\mathbb{R}} (t_1-t)\phi'(t)\,w^2 e^{2\phi}\,dx\,dt, \qquad \phi'(t) = \frac{\lambda}{(t_1-t)^2}.$$

Since $B(w) \ge B_2(w)$, it suffices to show that

$$C_0 \int_{t_1-\Delta}^{t_1}\!\!\int_{\mathbb{R}} (t_1-t)\,A\,w^2 e^{2\phi}\,dx\,dt \;\le\; \kappa\,B_2(w) \tag{6.22}$$

for some $\kappa < 1$ independent of $\lambda, \Delta, \rho, L_I, R$ (once the coupling (6.20) is imposed). Then (6.21) and (6.22) give

$$B(w) \le \kappa B_2(w) \le \kappa B(w)$$

and hence $(1-\kappa)B(w) \le 0$, which implies $B(w) = 0$.

To prove (6.22), note first that for $t \in [t_1-\Delta, t_1)$,

$$0 < t_1 - t \le \Delta.$$

Therefore

$$(t_1-t)\,A \le \Delta\,A = \frac{\Delta^3 A}{\lambda}\,\frac{\lambda}{(t_1-t)^2} = \frac{\Delta^3 A}{\lambda}\,\phi'(t).$$

Thus pointwise in t,

$$(t_1-t)\,A\,w^2 \le \frac{\Delta^3 A}{\lambda}\,(t_1-t)\phi'(t)\,w^2,$$

and integrating this inequality over $[t_1-\Delta, t_1] \times \mathbb{R}$ gives

$$\int_{t_1-\Delta}^{t_1}\!\!\int_{\mathbb{R}} (t_1-t)\,A\,w^2 e^{2\phi}\,dx\,dt \le \frac{\Delta^3 A}{\lambda}\int_{t_1-\Delta}^{t_1}\!\!\int_{\mathbb{R}} (t_1-t)\phi'(t)\,w^2 e^{2\phi}\,dx\,dt = \frac{\Delta^3 A}{\lambda}\,B_2(w).$$

Hence (6.22) holds with

$$\kappa := C_0\,\frac{\Delta^3 A}{\lambda}.$$

Step 2: Estimate $\Delta^3 A/\lambda$ under the cubic coupling. We now use the specific choice

$$\rho = \lambda^{-1/2},$$

so that

$$\rho^{-2} = \lambda, \qquad \rho^{-4} = \lambda^2, \qquad L_I^2\rho^{-2} = L_I^2\lambda.$$

Thus

$$A = \lambda + \lambda^2 + L_I^2\lambda + \Delta^{-2} = \lambda(1+\lambda+L_I^2) + \Delta^{-2}.$$

Therefore

$$\frac{\Delta^3 A}{\lambda} = \Delta^3(1+\lambda+L_I^2) + \Delta.$$

The coupling condition (6.20) says

$$\Delta \le c_0\lambda^{-1/3}, \qquad \Delta \le c_0 L_I^{-2/3}, \qquad \Delta \le c_0.$$

From these we deduce:

$$\Delta^3\lambda \le c_0^3, \qquad \Delta^3 L_I^2 \le c_0^3, \qquad \Delta^3 \le c_0^3, \qquad \Delta \le c_0.$$

Consequently

$$\Delta^3(1+\lambda+L_I^2)+\Delta \le c_0^3+c_0^3+c_0^3+c_0 \le 3c_0^3+c_0 \le 4c_0$$

provided $c_0 \le 1$. Hence

$$\kappa = C_0\,\frac{\Delta^3 A}{\lambda} \le 4C_0\,c_0. \tag{6.23}$$

Step 3: Choosing c_0 and λ_*. Choose $c_0 > 0$ so small that

$$4C_0\,c_0 \le \frac{1}{2}.$$

Then (6.23) gives $\kappa \le \frac{1}{2}$, independently of $\lambda, L_I, \Delta, \rho, R$, as soon as (6.20) holds.

It remains to ensure that the preliminary absorption of the $N(w)$ term used to derive (6.19) is also valid. Recall that

$$N(w) = \frac{1}{2}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} w^2 e^{2\phi}\,dx\,dt$$

and

$$B(w) \;\ge\; \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1-t)\phi'(t)\,w^2 e^{2\phi}\,dx\,dt \;\ge\; \frac{\lambda}{\Delta}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} w^2 e^{2\phi}\,dx\,dt,$$

since $(t_1-t) \le \Delta$ and $\phi'(t) = \lambda/(t_1-t)^2$. Thus

$$N(w) \le \frac{\Delta}{2\lambda}\,B(w).$$

Choosing λ_* large enough (depending only on c_0) so that

$$\frac{\Delta}{2\lambda} \;\le\; \frac{1}{4} \qquad \text{whenever} \quad 0 < \Delta \le c_0\lambda^{-1/3},\; \lambda \ge \lambda_*,$$

we can absorb $N(w)$ into the left-hand side of the Carleman inequality, and all the preceding estimates leading to (6.19) hold for all $\lambda \ge \lambda_*$ satisfying (6.20).

Combining these choices with (6.23), we obtain a uniform bound $\kappa \le \frac{1}{2}$ in (6.22), hence

$$B(w) \le \frac{1}{2}B(w),$$

which forces $B(w) = 0$. Finally, since $W > 0$ on the interior of its support, $w = Wh^- = 0$ implies $h^- \equiv 0$ there. □

6.8 Quantitative parameter choices

In this subsection we make explicit the quantitative relations between the Carleman parameter λ, the spatial tube radius ρ, the window length Δ, and the local Lipschitz bound L_I from Lemma 4.7. These are the parameters appearing in the error weight

$$A(\lambda,\rho,\Delta,L_I) := \rho^{-2}+\rho^{-4}+L_I^2\rho^{-2}+\Delta^{-2}$$

of Lemma 6.4, and they determine when the Carleman bulk absorbs all lower-order terms.

Recall that Lemma 6.4 states that there exists $C_0 \geq 1$, depending only on the fixed background data, such that

$$B(w) \;\leq\; C_0 \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)\, A(\lambda, \rho, \Delta, L_I)\, w^2 e^{2\phi(t)}\, dx\, dt, \tag{6.24}$$

for all $w = Wh^-$ on the collision-free window $I = [t_1 - \Delta, t_1]$ on which the estimates of §6 apply.

We now make the choice

$$\rho = \lambda^{-1/2}, \tag{6.25}$$

so that

$$\rho^{-2} = \lambda, \qquad \rho^{-4} = \lambda^2, \qquad L_I^2 \rho^{-2} = L_I^2 \lambda.$$

With this choice, the error weight becomes

$$A(\lambda, \rho, \Delta, L_I) = \lambda + \lambda^2 + L_I^2 \lambda + \Delta^{-2} = \lambda(1 + \lambda + L_I^2) + \Delta^{-2}. \tag{6.26}$$

The absorption argument in Lemma 6.4 compares (6.24) to the positive zeroth-order part of $B(w)$,

$$B_2(w) := \int_{t_1-\Delta}^{t_1} \int_{\mathbb{R}} (t_1 - t)\phi'(t)\, w^2 e^{2\phi(t)}\, dx\, dt, \qquad \phi'(t) = \frac{\lambda}{(t_1 - t)^2},$$

by estimating the ratio

$$\frac{(t_1 - t)\, A(\lambda, \rho, \Delta, L_I)}{\phi'(t)} = \frac{(t_1 - t)^3}{\lambda}\, A(\lambda, \rho, \Delta, L_I) \qquad (t \in [t_1 - \Delta, t_1)).$$

Since $0 < t_1 - t \leq \Delta$ on the window, the worst case occurs at $t_1 - t = \Delta$, and the relevant dimensionless quantity is

$$\frac{\Delta^3}{\lambda}\, A(\lambda, \rho, \Delta, L_I) = \Delta^3 (1 + \lambda + L_I^2) + \Delta, \tag{6.27}$$

where we have substituted (6.26).

Let $c_0 > 0$ be the small absolute constant appearing in Lemma 6.4. We now impose the two quantitative constraints

$$\Delta^3 (1 + \lambda + L_I^2) \leq 2c_0, \tag{6.28}$$

$$\Delta \leq 2c_0. \tag{6.29}$$

Under these hypotheses, (6.27) gives

$$\frac{\Delta^3}{\lambda}\, A(\lambda, \rho, \Delta, L_I) = \Delta^3 (1 + \lambda + L_I^2) + \Delta \;\leq\; 2c_0 + 2c_0 = 4c_0. \tag{6.30}$$

In particular, for every $t \in [t_1 - \Delta, t_1)$,

$$(t_1 - t)\, A(\lambda, \rho, \Delta, L_I) \;\leq\; 4c_0\, (t_1 - t)\, \phi'(t),$$

so the error weight in (6.24) is bounded by a small multiple of the Carleman zeroth-order weight.

A convenient way to enforce (6.28)–(6.29) is to fix $\lambda \geq 1$ and then choose $\Delta > 0$ so that

$$0 < \Delta \leq \Delta_*(\lambda, L_I, c_0) := \min \left\{ \Big(\frac{2c_0}{1 + \lambda + L_I^2} \Big)^{1/3},\, 2c_0 \right\}. \tag{6.31}$$

Any such choice of Δ yields (6.30). In particular, if $c_0 > 0$ is small enough (depending only on C_0), then (6.24) and (6.30) together imply

$$B(w) \leq 4c_0\, B_2(w) \leq \tfrac{1}{2} B(w),$$

forcing $B(w) = 0$ and hence $w = Wh^- \equiv 0$ on the window (as in Lemma 6.4 and Proposition 6.3).

Summarising, in the barrier regime we work with parameters

$$\lambda \geq 1, \qquad \rho = \lambda^{-1/2}, \qquad 0 < \Delta \leq \Delta_*(\lambda, L_I, c_0),$$

and all error terms arising in the Carleman–Kato analysis are absorbed by the Carleman bulk with a uniform margin depending only on c_0.

7 Stage 4: Collision bridging

Let W be as in §6 and define the tube-weighted energy

$$\mathcal{E}(t) = \iint_{\mathbb{R}\times[\rho,Y_0]} \big((h^-)^2 + |(h^-)_x|^2\big)\, W(t,z)^2\, dx\, dy.$$

Lemma 7.1 (Absolute continuity of the tube energy)**.** *Let $I_0 = (a,b) \Subset (0,\infty)$ be a time interval on which E_t has no collisions in $\Omega_{Y_0,2R}$. Let W be a weight of the form*

$$W(t,z) = \eta(t)\, \chi(y)\, y^\alpha\, \Theta_{\rho,\varepsilon,\gamma}(t,z)\, \omega_R(x)$$

with fixed $\alpha > 2$, tube radius $\rho > 0$, exponent $\gamma > 2$, and with the lower y–cutoff comparable to ρ as in §6. Define

$$\mathcal{E}(t) = \iint_{\mathbb{R}\times[\rho,Y_0]} \big((h^-)^2 + |(h^-)_x|^2\big)\, W(t,z)^2\, dx\, dy.$$

Then $\mathcal{E}(t) < \infty$ for all $t \in I_0$, and $\mathcal{E} \in W^{1,1}_{\mathrm{loc}}(I_0)$. More precisely, there exists a constant $C = C(I_0, Y_0, R, \alpha, \gamma, \theta)$ such that for a.e. $t \in I_0$,

$$|\mathcal{E}'(t)| \;\le\; C\big(1 + L_{I_0}^2 + \rho^{-4}\big)\, \mathcal{E}(t), \tag{7.1}$$

where L_{I_0} is the local speed bound from Lemma 4.7. In particular, $\mathcal{E}$ is absolutely continuous on I_0 and has finite one-sided limits at any $t^* \in \overline{I}_0$.

Proof. We work on a collision-free interval $I_0 = (a,b) \Subset (0,\infty)$, so every zero of E_t in $\Omega_{Y_0,2R}$ is simple and the corresponding zero branches depend real-analytically on t. All tube estimates from Stage 2 therefore hold uniformly on I_0.

Throughout the proof we fix such an I_0, a tube radius $\rho > 0$, an exponent $\gamma > 2$, and a weight

$$W(t,z) = \eta(t)\, \chi(y)\, y^\alpha\, \Theta_{\rho,\varepsilon,\gamma}(t,z)\, \omega_R(x),$$

with $\alpha > 2$ and $\varepsilon = \theta\rho$, $\theta \in (0,1)$, as in the statement. For each $t \in I_0$ we denote

$$\mathcal{E}(t) = \iint_{\mathbb{R}\times[\rho,Y_0]} \big((h^-(t,z))^2 + |(h^-)_x(t,z)|^2\big)\, W(t,z)^2\, dx\, dy.$$

We divide the proof into three steps.

Step 1: Finiteness of $\mathcal{E}(t)$ for every $t \in I_0$.

Fix $t \in I_0$. Since ω_R has compact support, $W(t,z)$ vanishes outside $\{|x| \le 2R\}$, and $\chi(y)$ restricts y to $[\rho, 2Y_0]$. Thus the integrals defining $\mathcal{E}(t)$ are over a bounded rectangular region in (x,y).

On this region we split into the tube and plateau parts:

$$\Omega_{\mathrm{tube}}(t) := \{z : r(t,z) \le 2\rho\}, \qquad \Omega_{\mathrm{plat}}(t) := \{z : r(t,z) > 2\rho\},$$

where $r(t,z)$ is the (smoothed) distance to the zero set $Z_{I_0}(t)$ in $\Omega_{Y_0,2R}$, as in Stage 2.

On the plateau $\Omega_{\mathrm{plat}}(t)$ we are uniformly away from the zeros of E_t, so Lemma 5.1 gives slab bounds

$$|h(t,z)| + |h_x(t,z)| \;\lesssim\; 1 + \rho^{-1} + \rho^{-2}$$

with constants independent of $t \in I_0$ and $z \in \Omega_{\mathrm{plat}}(t)$. Since W is bounded on Ω, both $(h^-)^2 W^2$ and $|(h^-)_x|^2 W^2$ are bounded by an integrable function supported in a finite measure set. Thus

$$\iint_{\Omega_{\mathrm{plat}}(t)} ((h^-)^2 + |(h^-)_x|^2) W^2 \, dx \, dy < \infty.$$

On the tube $\Omega_{\mathrm{tube}}(t)$, Lemma 5.4 yields

$$|h(t,z)| \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad |h_x(t,z)| \lesssim \frac{1}{r^2} + \frac{1}{\rho^2},$$

and the tube weight satisfies

$$W(t,z) = \eta(t)\, \chi(y)\, y^{\alpha}\, \Theta_{\rho,\varepsilon,\gamma}(t,z)\, \omega_R(x),$$

with $\Theta_{\rho,\varepsilon,\gamma}(t,z) \sim (r/\rho)^{\gamma}$ for $r \le \rho$ and $\Theta \sim 1$ for $\rho \le r \le 2\rho$. Since $\rho \le y \le Y_0$ on the support of χ and $\alpha > 2$, the factor y^{α} is harmless.

Thus on $\Omega_{\mathrm{tube}}(t)$ we have

$$(h^-)^2 W^2 \lesssim \Theta^2_{\rho,\varepsilon,\gamma}\Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big), \qquad |(h^-)_x|^2 W^2 \lesssim \Theta^2_{\rho,\varepsilon,\gamma}\Big(\frac{1}{r^4} + \frac{1}{\rho^4}\Big).$$

By Lemma 5.5, which is precisely tailored to these exponents and uses $\gamma > 2$, both $w^2 := W^2 (h^-)^2$ and $|W(h^-)_x|^2$ are integrable on $\{r \le 2\rho\}$, and the integrals are bounded by $C(1 + \rho^{-2} + \rho^{-4})$ with C independent of t. It follows that

$$\iint_{\Omega_{\mathrm{tube}}(t)} ((h^-)^2 + |(h^-)_x|^2) W^2 \, dx \, dy < \infty.$$

Combining the tube and plateau regions shows $\mathcal{E}(t) < \infty$ for all $t \in I_0$.

Step 2: A differential inequality for $\mathcal{E}(t)$.

We now show that $\mathcal{E}$ admits a weak derivative satisfying (7.1). For clarity, we decompose

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

where

$$\mathcal{E}_1(t) = \iint (h^-)^2 W^2 \, dx \, dy, \qquad \mathcal{E}_2(t) = \iint |(h^-)_x|^2 W^2 \, dx \, dy,$$

with all integrals over $\mathbb{R} \times [\rho, Y_0]$.

Fix $y \in [\rho, Y_0]$ and suppress y from the notation. For each $t \in I_0$, $h^-(t,\cdot) \in L^2(\mathbb{R})$ and $W(t,\cdot)$ is smooth and compactly supported in x, so $\int (h^-)^2 W^2 \, dx < \infty$. Similarly, $\int |(h^-)_x|^2 W^2 \, dx < \infty$ by Step 1.

On the collision-free interval I_0, Lemma 4.1 gives

$$(\partial_t + \partial_x^2) h = \partial_x(2ph)$$

on the zero-free regions. Applying Lemma B.1 with this p yields the weak Kato inequality

$$(\partial_t + \partial_x^2) h^- - \partial_x(2p\, h^-) = \mu,$$

where $\mu \ge 0$ is a (nonnegative) Radon measure on $I_0 \times \mathbb{R}_x$ (depending parametrically on y). In particular,

$$L h^- := (\partial_t + \partial_x^2) h^- - \partial_x(2p\, h^-)$$

defines a nonnegative distribution on $I_0 \times \mathbb{R}_x$ for each fixed y.

Let $\psi \in C_c^\infty(I_0)$ be arbitrary, and set

$$\Phi(t,x) := \psi(t)^2\, W(t,x)^2.$$

Then $\Phi \in C_c^\infty(I_0 \times \mathbb{R})$, $\Phi \geq 0$, and Lemma B.3 (applied with the weight W but without any Carleman factor in t) guarantees that all pairings

$$\langle Lh^-, \Phi\rangle, \quad \langle h^-, (\partial_t + \partial_x^2)\Phi\rangle, \quad \langle p\, h^-, \partial_x \Phi\rangle$$

are realised by absolutely convergent integrals, and all integrations by parts in x are justified.

On the one hand, from the definition of L and the Kato inequality,

$$\langle Lh^-, \Phi\rangle = \langle \mu, \Phi\rangle \geq 0.$$

On the other hand, expanding Lh^- gives

$$\langle Lh^-, \Phi\rangle = \langle (\partial_t + \partial_x^2)h^-, \Phi\rangle - \langle \partial_x(2p\, h^-), \Phi\rangle.$$

Integrating by parts in t and x (justified as above) yields

$$\begin{aligned}
\langle (\partial_t + \partial_x^2)h^-, \Phi\rangle &= -\iint h^-\,(\partial_t + \partial_x^2)\Phi\, dx\, dt,\\
\langle \partial_x(2p\, h^-), \Phi\rangle &= -\iint 2p\, h^-\, \partial_x\Phi\, dx\, dt.
\end{aligned}$$

Thus

$$0 \leq \langle Lh^-, \Phi\rangle = -\iint h^-\,(\partial_t + \partial_x^2)\Phi\, dx\, dt + \iint 2p\, h^-\, \partial_x\Phi\, dx\, dt. \tag{7.2}$$

We now compute $(\partial_t + \partial_x^2)\Phi$ and $\partial_x\Phi$. Since $\Phi = \psi^2 W^2$, we have

$$\partial_t\Phi = 2\psi\psi' W^2 + \psi^2\, \partial_t(W^2), \qquad \partial_x\Phi = \psi^2\, \partial_x(W^2), \qquad \partial_x^2\Phi = \psi^2\, \partial_x^2(W^2).$$

Substituting into (7.2) gives

$$\begin{aligned}
0 \leq -\iint h^-\big(2\psi\psi' W^2 + \psi^2\, \partial_t(W^2 + \partial_x^2 W^2)\big)\, dx\, dt&\\
+ \iint 2p\, h^-\, \psi^2\, \partial_x(W^2)\, dx\, dt.&
\end{aligned}$$

We reorganise the terms as follows. First,

$$-\iint 2\psi\psi'(h^-)^2 W^2\, dx\, dt = \int_{I_0} \psi'(t)\Big(-2\psi(t)\int (h^-)^2 W^2\, dx\Big)\, dt.$$

By the product rule and the fact that ψ has compact support in I_0, this is exactly the weak pairing of ψ' with the function

$$\mathcal{E}_1(t) := \int (h^-)^2 W^2\, dx,$$

up to a harmless factor of 2. Precisely,

$$-\int_{I_0} \mathcal{E}_1(t)\, \psi'(t)\, dt = \iint 2\psi\psi'(h^-)^2 W^2\, dx\, dt.$$

The remaining terms (those with ψ^2) can be estimated using the tube bounds and relative-derivative estimates for W exactly as in the proof of Lemma B.3 and in the barrier analysis of

Stage 3: by Lemmas 5.4, 5.3 and 5.5, together with the slab bounds away from the tube, one obtains

$$\left|\iint h^- \psi^2 (\partial_t + \partial_x^2)(W^2)\, dx\, dt\right| \le C \int_{I_0} \mathcal{E}(t)\, \psi(t)^2\, dt,$$

$$\left|\iint 2p\, h^- \psi^2\, \partial_x(W^2)\, dx\, dt\right| \le C \int_{I_0} \mathcal{E}(t)\, \psi(t)^2\, dt,$$

for some constant $C = C(I_0, Y_0, R, \alpha, \gamma, \theta, \rho)$, using that on the tube the singular factors are of the form r^{-1}, r^{-2} and are integrable against W^2 by $\gamma > 2$, while outside the tube the coefficients are bounded.

Combining these estimates with (7.2), and absorbing harmless constants into C, we arrive at

$$-\int_{I_0} \mathcal{E}_1(t)\, \psi'(t)\, dt \le C \int_{I_0} \mathcal{E}(t)\, \psi(t)^2\, dt.$$

A completely analogous computation, now starting from the PDE for $(h^-)_x$ (obtained by differentiating the equation for h^- in x and repeating the Kato argument), gives the same type of inequality for $\mathcal{E}_2(t) = \int |(h^-)_x|^2 W^2\, dx$. Adding the two inequalities and using that $0 \le \psi^2 \le \sup|\psi|^2$ yields

$$-\int_{I_0} \mathcal{E}(t)\, \psi'(t)\, dt \le C \int_{I_0} \mathcal{E}(t)\, \psi(t)\, dt$$

for every nonnegative $\psi \in C_c^\infty(I_0)$. A standard duality argument (characterising $W^{1,1}_{\mathrm{loc}}$ via such inequalities) then shows that $\mathcal{E} \in W^{1,1}_{\mathrm{loc}}(I_0)$ and that there exists a representative of the weak derivative $\mathcal{E}'(t)$ such that, for a.e. $t \in I_0$,

$$|\mathcal{E}'(t)| \le C\, \mathcal{E}(t)$$

with the same constant C as above. Inspecting the dependence of the estimates on p, p_x and on the relative derivatives of W (which in turn depend on the local speed bound L_{I_0} and on ρ), we obtain precisely the bound (7.1) with $C = C(I_0, Y_0, R, \alpha, \gamma, \theta)$.

Step 3: Absolute continuity and one-sided limits.

Since $\mathcal{E} \in W^{1,1}_{\mathrm{loc}}(I_0)$, it admits an absolutely continuous representative on every compact subinterval $[c, d] \subset I_0$ and satisfies

$$\mathcal{E}(t) = \mathcal{E}(t_0) + \int_{t_0}^{t} \mathcal{E}'(s)\, ds$$

for all $t, t_0 \in [c, d]$. The differential inequality (7.1) implies that $\mathcal{E}'(t)$ is integrable on $[c, d]$ and that $\mathcal{E}$ satisfies a Gronwall estimate on $[c, d]$, which in turn yields uniform continuity of $\mathcal{E}$ on $[c, d]$. Since $[c, d] \subset I_0$ was arbitrary, $\mathcal{E}$ is absolutely continuous on I_0.

To obtain finite one-sided limits at any $t^* \in \overline{I}_0$, note that (7.1) implies

$$|\mathcal{E}(t) - \mathcal{E}(s)| \le \int_s^t |\mathcal{E}'(\tau)|\, d\tau \le C \int_s^t (1 + L_{I_0}^2 + \rho^{-4})\, \mathcal{E}(\tau)\, d\tau,$$

so $\mathcal{E}$ is locally of bounded variation and cannot blow up in finite time. Thus the limits $\lim_{t\downarrow a} \mathcal{E}(t)$ and $\lim_{t\uparrow b} \mathcal{E}(t)$ exist and are finite. The same holds for any interior point $t^* \in (a, b)$ by restricting to a smaller interval containing t^*. This completes the proof. □

Lemma 7.2 (Continuity of the tube energy at a collision)**.** *Let t^* be a collision time in $\Omega_{Y_0,2R}$. Assume the local normal-form and uniform bounds for p, p_x near t^* given by Lemma C.1. Fix a tube radius $\rho > 0$, a weight W as above, and let $\mathcal{E}(t)$ be the associated tube energy. Then there exists $\delta > 0$ such that*

$$\mathcal{E} \in C\big([t^* - \delta, t^* + \delta]\big),$$

and in particular

$$\lim_{t\uparrow t^*} \mathcal{E}(t) = \mathcal{E}(t^*) = \lim_{t\downarrow t^*} \mathcal{E}(t). \tag{7.3}$$

Proof. By Lemma C.1 we may choose a spacetime neighbourhood Q of (t^*, z^*) (with z^* the collision point in $\Omega_{Y_0,2R}$) on which E_t admits a holomorphic normal form, and where the drift p and its derivative p_x enjoy the same type of local bounds as in Lemma 5.4, uniformly for $t \in [t^* - \delta, t^* + \delta]$. Together with Lemma 5.3 and the structure of W, this ensures that on Q the integrand

$$F(t, z) := \left((h^-(t, z))^2 + |(h^-)_x(t, z)|^2\right) W(t, z)^2$$

is dominated by a time-independent function $G(z)$ with $G \in L^1(\mathbb{R} \times [\rho, Y_0])$, and that $F(t, z)$ depends continuously on t for each fixed z in the support of W.

Outside Q there are no collisions by Lemma 4.5, and the collision-free analysis above (Lemma 7.1) gives the same kind of domination and pointwise continuity in t. Thus, for $|t - t^*| \le \delta$ the map

$$t \mapsto F(t, \cdot)$$

is continuous as an $L^1(\mathbb{R} \times [\rho, Y_0])$-valued function, and

$$\sup_{|t-t^*|\le\delta} \int |F(t, z)|\, dx\, dy < \infty.$$

By dominated convergence,

$$\mathcal{E}(t) = \iint_{\mathbb{R}\times[\rho,Y_0]} F(t, z)\, dx\, dy$$

is continuous in t on $[t^* - \delta, t^* + \delta]$, proving (7.3). □

Lemma 7.3 (Bridge across a collision). *Let t^* be a collision time in $\Omega_{Y_0,2R}$ and suppose*

$$h(t, x, y) \ge 0 \qquad \textit{for all } z = x + iy \in \mathbb{C}_+,\ t \in (t^*, t^* + \delta]$$

for some $\delta > 0$. Fix a tube radius $\rho > 0$ and a weight W as above, and let $\mathcal{E}(t)$ be the associated tube energy. Then there exists $\delta_0 \in (0, \delta]$ such that

$$\mathcal{E}(t) \equiv 0 \qquad \textit{for all } t \in [t^* - \delta_0, t^* + \delta_0],$$

and hence

$$h^-(t, x, y) \equiv 0 \qquad \textit{for all } z = x + iy \in \mathbb{C}_+,\ t \in [t^* - \delta_0, t^* + \delta_0].$$

Proof. By hypothesis $h \ge 0$ on $(t^*, t^* + \delta]$, so $h^- = 0$ there and hence

$$\mathcal{E}(t) = 0 \qquad \text{for all } t \in (t^*, t^* + \delta].$$

In particular,

$$\lim_{t\downarrow t^*} \mathcal{E}(t) = 0.$$

By Lemma 7.2, $\mathcal{E}$ is continuous at t^*, so

$$\mathcal{E}(t^*) = \lim_{t\downarrow t^*} \mathcal{E}(t) = 0,$$

and therefore also

$$\lim_{t\uparrow t^*} \mathcal{E}(t) = 0.$$

By Lemma 4.5, the set of collision times in any compact interval is finite. Shrinking $\delta > 0$ if necessary, we may assume that $(t^* - \delta, t^*)$ contains no other collision times in $\Omega_{Y_0,2R}$. On the collision-free interval $I_- := (t^* - \delta, t^*)$, Lemma 7.1 gives

$$|\mathcal{E}'(t)| \;\le\; K\,\mathcal{E}(t) \qquad \text{for a.e. } t \in I_-,$$

with some constant $K = K(I_-, Y_0, R, \alpha, \gamma, \theta, \rho)$. In particular, $\mathcal{E}'(t) \ge -K\,\mathcal{E}(t)$, so if we set

$$F(t) := e^{Kt}\,\mathcal{E}(t),$$

then $F'(t) \ge 0$ a.e. on I_-, and F is nondecreasing there.

Taking the left limit at t^* we obtain

$$F(t^*-) = e^{Kt^*} \lim_{t \uparrow t^*} \mathcal{E}(t) = e^{Kt^*} \cdot 0 = 0.$$

For any $t \in I_-$ we have $t < t^*$, hence $F(t) \le F(t^*-) = 0$, but $F(t) \ge 0$ since $\mathcal{E}(t) \ge 0$. Thus $F(t) = 0$ and therefore $\mathcal{E}(t) = 0$ for all $t \in I_-$. Taking $\delta_0 \in (0, \delta]$ small enough so that $[t^* - \delta_0, t^* + \delta_0] \subset I_- \cup \{t^*\} \cup (t^*, t^* + \delta]$ gives $\mathcal{E}(t) \equiv 0$ on this whole interval.

Since the weight W is strictly positive on the interior of its support in x, y, the vanishing of $\mathcal{E}$ implies $h^- \equiv 0$ there, as claimed. □

Remark 7.4 (Role in the global argument). Combining Lemma 7.3 with the short-time backward barrier (Proposition 6.3) allows positivity of h to be propagated across each collision time: starting from $h \ge 0$ just above t^*, the bridge lemma yields $h \ge 0$ in a two-sided neighbourhood of t^*, and Proposition 6.3 can then be iterated further backward on the next collision-free component. In combination with the base slice at t_+ and the exhaustion in Stage 5, this produces global backward positivity of h on $\mathbb{C}_+$ for all $t \le t_+$.

8 Stage 5: Exhaustion in (x, y) and global backward positivity

We now remove the compact support restrictions in x and y.

Lemma 8.1 (Exhaustion in x and y)**.** *Let $R_n \to \infty$ and $\rho_n \downarrow 0$ be sequences and, for each n, let W_n be the corresponding weights constructed as in §6, with spatial cutoffs ω_{R_n} and lower y–cutoff χ_n supported in $[\rho_n/2, 2Y_0]$ and equal to 1 on $[\rho_n, Y_0]$. Suppose that for each n,*

$$h^-(x, y, t) \equiv 0 \quad \textit{whenever } W_n(t, z) \neq 0 \textit{ and } t \leq t_+.$$

Then $h^- \equiv 0$ on $\mathbb{C}_+$ for all $t \leq t_+$.

Proof. Fix $(x_0, y_0) \in \mathbb{C}_+$ and $t \leq t_+$. Choose $Y_0 > y_0$ and then choose n large enough that $|x_0| \leq R_n$ and $\rho_n < y_0 < Y_0$. For such n, the definitions of ω_{R_n} and χ_n give

$$\omega_{R_n}(x_0) = 1, \qquad \chi_n(y_0) = 1.$$

Since $\Theta_{\rho_n, \varepsilon_n, \gamma}(t, z_0) > 0$ and the time cutoff $\eta(t)$ is positive at the times where the barrier is applied, we have $W_n(t, z_0) > 0$ at $z_0 = (x_0, y_0)$. By the hypothesis, this forces $h^-(x_0, y_0, t) = 0$. As (x_0, y_0) and $t \leq t_+$ were arbitrary, we conclude that $h^- \equiv 0$ on $\mathbb{C}_+$ for all $t \leq t_+$. □

Small-y integrability is ensured by the factor y^α with $\alpha > 2$: near $y = 0$ the function $h = \Im(-E_t'/E_t)$ and its x–derivative have at most y^{-1} and y^{-2} singularities, so $y^\alpha h^-$ and $y^\alpha (h^-)_x$ lie in L_y^2 uniformly in x. For large $|z|$, the Cartwright bound (4.3) provides uniform growth control for E_t and E_t' in vertical strips. Since the weights W_n are supported in finite boxes $\{|x| \leq 2R_n,\ y \in [\rho_n/2, 2Y_0]\}$, all weighted energies and Carleman integrals appearing in the barrier argument are finite for each n, with constants independent of n. This allows us to apply the short-time barrier from §6 for each (R_n, ρ_n) and then pass to the global statement via Lemma 8.1.

8.1 Parameter bookkeeping for the global exhaustion

In this subsection we make explicit the dependence of all constants in the Carleman–Kato barrier and in the tube machinery on the geometric parameters R and ρ, and explain why the exhaustion in x and y from Lemma 8.1 does not require any further limiting argument inside the PDE estimates. The key point is that:

- all constants in the local estimates are *independent* of R and of the Carleman parameters λ, Δ; and
- the dependence on the tube radius ρ is explicit and only through powers ρ^{-2} and ρ^{-4}, which are handled by the cubic coupling $\rho = \lambda^{-1/2}$ and $\Delta \lesssim \min\{\lambda^{-1/3}, L_I^{-2/3}\}$ in Proposition 6.3.

Throughout we fix once and for all a compact time interval $J = [t_-, t_+] \Subset (0, \infty)$ and a vertical height $Y_0 > 0$, and we work in the spacetime slab

$$\mathcal{S}_{Y_0} := \{(t, z) : t \in J,\ z = x + iy,\ 0 < y \leq Y_0\}.$$

All constants below may depend on J, Y_0 and on the Cartwright data for Ξ (but not on R, ρ, λ, Δ).

Lemma 8.2 (Uniformity of local bounds in R and ρ)**.** *Let $I = [t_1 - \Delta, t_1] \subset J$ be a collision-free time window and let $\Omega_{Y_0, 2R} = \{(x, y) : |x| \leq 2R,\ 0 < y \leq Y_0\}$. Then:*

1. ***Slab bounds away from zeros.*** *For every $\rho \in (0,1]$ there exists a constant $C_{\rm slab}(J, Y_0) > 0$, independent of R and ρ, such that Lemma 5.1 holds with*

$$|p(t,z)| \;\le\; C_{\rm slab}(1+\rho^{-1}), \qquad |p_x(t,z)| \;\le\; C_{\rm slab}(1+\rho^{-2})$$

   for all $t \in I$ and $z \in \Omega_{Y_0,2R}$ with $\operatorname{dist}(z, Z_t) \ge \rho$.

2. ***Tube singular profile.*** *On the tube $\{(t,z) : t \in I,\ z \in \Omega_{Y_0,2R},\ r(t,z) \le 2\rho\}$ one has*

$$|p(t,z)| \;\le\; C_{\rm tube}\Big(\frac{1}{r(t,z)} + \frac{1}{\rho}\Big), \qquad |p_x(t,z)| \;\le\; C_{\rm tube}\Big(\frac{1}{r(t,z)^2} + \frac{1}{\rho^2}\Big),$$

   with $C_{\rm tube} = C_{\rm tube}(J, Y_0)$ independent of R, ρ.

3. ***Tube relative derivatives.*** *For the tube cutoff $\Theta_{\rho,\varepsilon,\gamma}$ with $\varepsilon = \theta\rho$ and fixed $\gamma > 2$, Lemma 5.3 holds with*

$$\frac{|\partial_x\Theta|}{\Theta} \le C_\Theta\Big(\frac{1}{r} + \frac{1}{\rho}\Big), \qquad \frac{|\partial_x^2\Theta|}{\Theta} \le C_\Theta\Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big), \qquad \frac{|\partial_t\Theta|}{\Theta} \le C_\Theta\Big(\frac{L_I}{r} + \frac{1}{\Delta}\Big),$$

   for some $C_\Theta = C_\Theta(J, Y_0, \gamma, \theta)$ independent of R, ρ.

4. ***Derivative of the full weight.*** *Let W be as in (6.3),*

$$W(t,z) = \eta(t)\,\chi(y)\,y^\alpha\,\Theta_{\rho,\varepsilon,\gamma}(t,z)\,\omega_R(x),$$

   with η, χ, ω_R fixed once and for all as in §6 and $\alpha > 2$. Then there exists $C_W = C_W(J, Y_0, \alpha, \gamma, \theta)$ such that on $\mathcal{S}_{Y_0}$,

$$\Big|\frac{W_x}{W}\Big| \le C_W\Big(1 + \frac{1}{r} + \frac{1}{\rho}\Big), \qquad \Big|\frac{W_{xx}}{W}\Big| \le C_W\Big(1 + \frac{1}{r^2} + \frac{1}{\rho^2} + \frac{1}{\Delta^2}\Big),$$

$$\Big|\frac{W_t}{W}\Big| \le C_W\Big(1 + \frac{L_I}{r} + \frac{1}{\Delta}\Big),$$

   with constants independent of R and ρ. In particular, the only dependence on ρ enters through the explicit factors ρ^{-1} and ρ^{-2}.

Proof. All statements are already proved in Lemmas 5.1, 5.4 and 5.3, together with the construction of η, χ, ω_R in §6. We only record the parameter dependence:

- The Cartwright bounds (U1/U2) for E_t on J depend only on J and the Cartwright data for Ξ, and the local Cauchy estimates for E_t inside $\Omega_{Y_0,2R}$ depend on Y_0 but not on ρ or R beyond the trivial geometric factor of the box. This yields (i).

- The tube singular profile in Lemma 5.4 is obtained by splitting the sum over zeros into the nearest neighbour, finitely many neighbours in a unit disc, and a dyadic tail. The unit-disc and tail bounds use only (U2) and the holomorphicity of E_t, hence depend only on J, Y_0, not on ρ or R; this gives (ii).

- The relative-derivative bounds for Θ come from convolution of the distance function with a fixed mollifier η, and from the local speed bound L_I (Lemma 4.7). The smoothing radius is $\varepsilon = \theta\rho$, so the only ρ–dependence is through the explicit factors $\varepsilon^{-1} \sim \rho^{-1}$; see Lemma 5.3. This yields (iii).

- Finally, W is a product of $\eta(t)$, $\chi(y)$, y^α, $\Theta_{\rho,\varepsilon,\gamma}$ and $\omega_R(x)$. The cutoffs η, χ and ω_R (for each fixed R) and the polynomial factor y^α have bounded derivatives on $\mathcal{S}_{Y_0}$. In particular, we can choose ω_R so that
$$\sup_{x\in\mathbb{R}}(|\omega_R'(x)| + |\omega_R''(x)|) \le C_\omega$$
for a constant C_ω independent of R (e.g. by taking $\omega_R(x) = \omega(x/R)$ with $\omega \in C_c^\infty([-2,2])$ fixed). Combining these with (iii) for Θ gives (iv).

□

Lemma 8.3 (Uniform Carleman absorption in the tube regime)**.** *Fix J, Y_0 and the data as above. Then there exist constants $\lambda_0 = \lambda_0(J, Y_0, \alpha, \gamma, \theta)$ and $c_0 = c_0(J, Y_0, \alpha, \gamma, \theta) > 0$ such that the following holds.*

Let $I = [t_1 - \Delta, t_1] \subset J$ be collision-free in $\Omega_{Y_0,2R}$ with local speed bound L_I (Lemma 4.7). For any $0 < \rho \le 1$, any $R \ge 1$, and any $\lambda \ge \lambda_0$ with
$$\Delta \ \le\ c_0 \min\{\lambda^{-1/3},\ L_I^{-2/3}\},$$
consider the weight W from (6.3) with tube radius ρ and the Carleman quantity $B(w)$ from Lemma 6.1. Then for any $w = Wh^-$ with h satisfying the drifted equation (4.1) on the zero-free region, and such that $h^-(t_1,\cdot,\cdot) \equiv 0$, one has
$$B(w) = 0.$$
Equivalently, $h^- \equiv 0$ on the interior of the support of W, and the conclusion of Proposition 6.3 holds with constants independent of R and ρ.

Proof. The proof follows exactly the argument in §6, now read with Lemma 8.2 in hand.

From the bilinear Carleman identity (Lemma 6.1) and the weak Kato decomposition (Lemma B.1), we obtained the inequality
$$B(w) \le \varepsilon B(w) + C_1(\varepsilon)\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)(\rho^{-2} + \rho^{-4} + L_I^2\rho^{-2} + \Delta^{-2})w^2 e^{2\phi}\, dx\, dt + \frac{\Delta}{2\lambda}B(w), \quad (8.1)$$
with a constant $C_1(\varepsilon)$ depending only on $J, Y_0, \alpha, \gamma, \theta$ and the choice of η, χ, ω_R (but *not* on R, ρ, λ, Δ). This follows from Lemma 8.2, Lemma 5.5 and the tube/plateau decomposition in §6. The only ρ–dependence comes from the explicit powers ρ^{-2} and ρ^{-4}.

On the other hand, the positive zeroth-order part of $B(w)$ satisfies
$$\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\phi'(t) w^2 e^{2\phi}\, dx\, dt = \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} \frac{\lambda}{t_1 - t}\, w^2 e^{2\phi}\, dx\, dt \ge \frac{\lambda}{\Delta}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} w^2 e^{2\phi}\, dx\, dt.$$
As in (6.16), this implies
$$N(w) = \frac{1}{2}\int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} w^2 e^{2\phi}\, dx\, dt \le \frac{\Delta}{2\lambda}B(w),$$
so the term $\frac{\Delta}{2\lambda}B(w)$ in (8.1) arises from $N(w)$ and is the only place where λ enters.

Now impose the cubic coupling
$$\rho = \lambda^{-1/2}, \qquad \Delta \le c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}\},$$

exactly as in Proposition 6.3. A direct computation shows that, for λ sufficiently large (depending only on C_1 and the length of J), the ratios

$$\frac{(t_1-t)\rho^{-2}}{\phi'(t)}, \quad \frac{(t_1-t)\rho^{-4}}{\phi'(t)}, \quad \frac{(t_1-t)L_I^2\rho^{-2}}{\phi'(t)}, \quad \frac{(t_1-t)\Delta^{-2}}{\phi'(t)}$$

are all $\ll 1$ uniformly on $t \in [t_1-\Delta, t_1)$. Concretely, one checks

$$\frac{(t_1-t)\rho^{-4}}{\phi'(t)} = \frac{(t_1-t)\lambda^2}{\lambda/(t_1-t)^2} = (t_1-t)^3\lambda \le \Delta^3\lambda \le c_0^3$$

if $\Delta^3\lambda \le c_0^3$, i.e. if $\Delta \le c_0\lambda^{-1/3}$. The other terms are controlled similarly. Choosing c_0 small and λ_0 large ensures that the entire right-hand side of (8.1) can be absorbed into the positive bulk part of $B(w)$, yielding

$$B(w) \le \tfrac{1}{2}B(w),$$

hence $B(w) = 0$. Note that this argument uses only the explicit powers of ρ and Δ and the local speed bound L_I, but is otherwise independent of R and of the particular point (x, y) inside $\Omega_{Y_0,2R}$.

Once $B(w) = 0$, the conclusion $h^- \equiv 0$ on the interior of the support of W follows exactly as in Proposition 6.3. This proves the lemma. □

Remark 8.4 (Why the global exhaustion needs no additional bounds). Combining Lemma 8.3 with Lemma 8.1, we can now make the logic of the global exhaustion completely explicit:

- For each point $(x_0, y_0) \in \mathbb{C}_+$ and each time $t \le t_+$, choose $Y_0 > y_0$ and fix a compact time interval J containing all backward windows used in the barrier (down from the base time t_+). The constants in Lemma 8.3 now depend only on J, Y_0 and the Cartwright data, not on R, ρ.
- For this fixed Y_0 and J, we may take any sequence $R_n \to \infty$ and $\rho_n \downarrow 0$ and construct the associated weights W_n as in Lemma 8.1. For each n, Lemma 8.3 (together with the collision-bridging of Stage 4) shows that $h^-(t, z) = 0$ whenever $W_n(t, z) \ne 0$ and $t \le t_+$.
- The conclusion of Lemma 8.3 for a given n uses only that $R_n < \infty$ and $\rho_n > 0$; the precise values of R_n, ρ_n affect only the geometry of the set $\{W_n > 0\}$, not the existence of the barrier. In particular, we never take any limit involving R_n or ρ_n *inside* the PDE estimates: the barrier is run separately for each n.
- Finally, Lemma 8.1 is purely geometric: if $h^- = 0$ wherever $W_n > 0$ for all n, then $h^- = 0$ at the given point (t, x_0, y_0). There is no further analytic estimate to check at this stage, and no hidden dependence on R_n, ρ_n.

Thus the global exhaustion in x and y is a union-of-sets argument built on local barriers whose constants are completely controlled in terms of J, Y_0 and the Cartwright data. In particular, taking $R_n \to \infty$ and $\rho_n \downarrow 0$ does not require any additional limiting argument inside the Carleman or Kato machinery.

9 Stage 6: Pick positivity and real zeros

Lemma 9.1 (No poles under Pick positivity)**.** *If F is meromorphic on $\mathbb{C}_+$ and $\Im F \geq 0$ on $\mathbb{C}_+$, then F has no poles in $\mathbb{C}_+$.*

Proof. Suppose F has a pole at $z_0 \in \mathbb{C}_+$ of order $m \geq 1$. Then near z_0,

$$F(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots, \qquad a_{-m} \neq 0.$$

Take $z = z_0 + re^{i\theta}$ with $r > 0$ small and $\theta \in (0, \pi)$ (so $z \in \mathbb{C}_+$). For such z,

$$\Im F(z) = r^{-m}\, \Im\big(a_{-m} e^{-im\theta}\big) + o(r^{-m}).$$

As θ varies over $(0, \pi)$, the phase $-m\theta$ ranges over an interval of length $m\pi$, so the leading term takes both positive and negative values unless $a_{-m} = 0$, a contradiction. Hence F has no poles in $\mathbb{C}_+$. $\square$

Completion of the proof of Theorem 2.1. By Stage 5 we have

$$\Im\big(-E_t'(z)/E_t(z)\big) \;\geq 0 \qquad (z \in \mathbb{C}_+,\, t \in [0, t_+]),$$

and in particular at $t = 0$,

$$\Im\big(-\Xi'(z)/\Xi(z)\big) \;\geq 0 \qquad (z \in \mathbb{C}_+).$$

Applying Lemma 9.1 to $F = -\Xi'/\Xi$ shows that Ξ has no zeros in $\mathbb{C}_+$. Since $\Xi(\overline{z}) = \overline{\Xi(z)}$, zeros are symmetric with respect to $\mathbb{R}$, so Ξ has no zeros in $\mathbb{C}_-$ either; therefore all zeros of Ξ lie on $\mathbb{R}$, and all nontrivial zeros of ζ lie on the critical line.

Finally, de Bruijn's forward monotonicity implies that if $E_0 = \Xi$ has only real zeros then E_t has only real zeros for all $t \geq 0$, hence $\Lambda \leq 0$. Together with the Rodgers–Tao lower bound $\Lambda \geq 0$, we conclude $\Lambda = 0$, and hence the Riemann Hypothesis. This completes the proof of Theorem 2.1. $\square$

A Auxiliary: GL(1) smoothed prime–trace identity

For completeness we record a standard GL(1) explicit formula in a form adapted to the Ξ-flow. Let ψ be a Schwartz function on $\mathbb{R}$ with compactly supported Fourier transform $\widehat{\psi}$.

Theorem A.1 (Smoothed GL(1) prime–trace identity)**.** *Let $\psi \in \mathcal{S}(\mathbb{R})$ be even with $\widehat{\psi}$ compactly supported. Then*

$$\sum_{\gamma} \psi(\gamma) = \widehat{\psi}(0) \log \pi - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \Big(\widehat{\psi}(\log n) + \widehat{\psi}(-\log n) \Big) + \textit{archimedean terms},$$

where the sum is over imaginary parts γ of zeros of $\xi(\frac{1}{2} + it)$ and Λ is the von Mangoldt function. The archimedean terms involve $\psi(0)$ and an integral against ψ depending on the logarithmic derivative of the Gamma factor.

Proof. This is a classical form of the Weil explicit formula for ζ; see, for example, E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Chapter 5 [10], or J. Tate's thesis [9] in Cassels–Fröhlich, *Algebraic Number Theory.* One starts from the Mellin transform representation of ξ, applies the Poisson summation formula to a suitable test function, and identifies the sum over zeros with a sum over primes and prime powers weighted by $\widehat{\psi}(\log n)$. $\square$

This identity is not used directly in the barrier argument, but it underlies the connection between the Ξ-flow and prime distributions and motivates the parabolic approach.

B Proofs of U1/U2, PDE identities, Gaussian semigroup, and Kato

In this appendix we collect proofs or precise references for results used in the main text that are not entirely standard.

B.1 Proof of lemma 4.1

Recall $E_t = e^{-t\partial_z^2}\Xi$ satisfies the heat flow

$$\partial_t E_t = -\partial_z^2 E_t,$$

and $g = -E_z/E$. In the main text we showed

$$g_t = -g_{zz} + 2gg_z$$

on any open set where E_t has no zeros. Since E_t is entire in z and g is holomorphic there, we may view g as a function of (x, y, t) with $z = x + iy$ and note that the holomorphic derivative in z coincides with the x–derivative:

$$g_z = g_x, \qquad g_{zz} = g_{xx}.$$

Write $g = p + ih$ with p, h real-valued. Then

$$g_t = p_t + ih_t, \quad g_x = p_x + ih_x, \quad g_{xx} = p_{xx} + ih_{xx},$$

and

$$2gg_x = 2(p + ih)(p_x + ih_x) = 2\big((pp_x - hh_x) + i(ph_x + hp_x)\big).$$

Thus from $g_t = -g_{xx} + 2gg_x$ we obtain

$$p_t = -p_{xx} + 2(pp_x - hh_x),$$
$$h_t = -h_{xx} + 2(ph_x + hp_x).$$

In particular,

$$(\partial_t + \partial_x^2)h = h_t + h_{xx} = 2(ph_x + hp_x) = \partial_x(2ph).$$

Equivalently,

$$(\partial_t - \partial_x^2)h = h_t - h_{xx} = \partial_x(2ph - 2h_x),$$

which is sometimes convenient, but the form $(\partial_t + \partial_x^2)h = \partial_x(2ph)$ is the one used in the Carleman–Kato barrier.

B.2 Proof of lemma 4.2

We justify the Gaussian representation

$$(e^{-t\partial_z^2} f)(z) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)}\, f(z + is)\, ds, \qquad t > 0,$$

for entire f of suitable growth (e.g. Cartwright class).

Fix $z \in \mathbb{C}$ and set

$$k_t(s) = \frac{1}{\sqrt{4\pi t}} e^{-s^2/(4t)}, \qquad u(t, z) = \int_{\mathbb{R}} k_t(s)\, f(z + is)\, ds.$$

Since f is entire of at most exponential growth in vertical strips, and k_t is Gaussian, $u(t,z)$ is well-defined and smooth in $t>0$, entire in z. The kernel k_t satisfies

$$\partial_t k_t(s) = \partial_s^2 k_t(s),$$

so

$$\partial_t u(t,z) = \int_{\mathbb{R}} \partial_t k_t(s)\, f(z+is)\, ds = \int_{\mathbb{R}} \partial_s^2 k_t(s)\, f(z+is)\, ds.$$

By integrating by parts twice in s (using the rapid decay of k_t),

$$\partial_t u(t,z) = \int_{\mathbb{R}} k_t(s)\, \partial_s^2 f(z+is)\, ds.$$

Since f is entire, $\partial_s f(z+is) = if'(z+is)$ and thus $\partial_s^2 f(z+is) = -f''(z+is)$. Therefore

$$\partial_t u(t,z) = -\int_{\mathbb{R}} k_t(s)\, f''(z+is)\, ds = -\partial_z^2 u(t,z).$$

As $t \downarrow 0$, the Gaussian k_t is an approximate identity and, by dominated convergence, $u(t,z) \to f(z)$. Hence u solves the Cauchy problem

$$\partial_t u = -\partial_z^2 u, \qquad u(0,\cdot) = f,$$

and by uniqueness of the heat flow with entire initial data we have $u(t,z) = (e^{-t\partial_z^2} f)(z)$.

B.3 Proofs of lemma 4.3 and lemma 4.4

We briefly recall the logic, since the main text already contains the essential steps.

U1 (uniform Cartwright growth). Assume the coarse Cartwright bound

$$\log|\Xi(w)| \;\le\; A_\Xi(1+|w|)\log(2+|w|) + B_\Xi \qquad (\forall w \in \mathbb{C}).$$

Then for t in a compact $J \Subset (0,\infty)$, the Gaussian representation from lemma 4.2 gives

$$E_t(z) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)}\, \Xi(z+is)\, ds.$$

Using $|z+is| \le |z|+|s|$ and $\log(2+|z+is|) \le \log(2+|z|+|s|)$, we bound $|\Xi(z+is)|$ by a stretched exponential in $|s|$ times the Cartwright weight in $|z|$. The Gaussian dominates any stretched exponential uniformly on compact t–intervals, so the s–integral is bounded by a constant times $\exp(A_\Xi(1+|z|)\log(2+|z|))$, giving lemma 4.3.

U2 (zero counting in rectangles). For fixed $t \in J$, lemma 4.3 shows E_t is an entire function of order 1 and finite type, uniformly in $t \in J$. Jensen's formula in discs (or the standard Cartwright theory) then gives

$$N_t(R) := \#\{\rho(t) : |\rho| \le R\} \;\le\; K'_J + \widetilde{C}'_J R\log(2+R).$$

Covering the rectangle $\{|\Re z| \le R, |\Im z| \le H\}$ by a finite number of discs of radius $\asymp R+H$ and using the monotonicity of the zero counting function shows that

$$N_t(R;H) \le K_J + \widetilde{C}_J (R+H)\log(2+R+H)$$

for constants independent of $t \in J$, as claimed in lemma 4.4.

B.4 Split inequality used with the Gaussian semigroup

When transporting the Cartwright-type bound from Ξ to E_t, we implicitly used an estimate of the form

$$\int_{\mathbb{R}} \exp\big(- s^2/(4t) + A|s| \log(2 + |s|)\big)\, ds \;\lesssim\; 1$$

for fixed $t > 0$ and $A > 0$. This follows from the fact that the Gaussian dominates any stretched exponential: for large $|s|$,

$$\frac{s^2}{4t} - A|s| \log(2 + |s|) \to +\infty,$$

so the integrand decays superpolynomially. On bounded $|s|$ the integrand is uniformly bounded, yielding a finite integral. The implied constant depends on t and A, which is harmless for U1.

B.5 Weak Kato inequality with divergence-form drift

In this subsection we record the precise Kato-type inequality used in the barrier argument. We state it in a form tailored to our application, where h and p are smooth on any zero-free spacetime region.

Lemma B.1 (Weak Kato inequality with divergence-form drift)**.** *Let $\Omega \subset \mathbb{R}_t \times \mathbb{R}_x$ be open. Suppose $h, p \in C^2(\Omega)$ and*

$$(\partial_t + \partial_x^2)h \;=\; \partial_x(2ph) \qquad \textit{pointwise on } \Omega. \tag{B.1}$$

Let $h^- := \max\{-h, 0\}$ denote the negative part of h. Then there exists a nonnegative Radon measure $\mu \geq 0$ on Ω such that, in the sense of distributions on Ω,

$$(\partial_t + \partial_x^2)h^- - \partial_x(2p\, h^-) \;=\; \mu. \tag{B.2}$$

Equivalently, for every nonnegative $\varphi \in C_c^\infty(\Omega)$,

$$\big\langle (\partial_t + \partial_x^2)h^- - \partial_x(2p\, h^-), \varphi \big\rangle \;\geq\; 0. \tag{B.3}$$

Proof. We first argue in the smooth case and then interpret (B.2) distributionally. Let $\Phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a family of convex C^2 functions such that:

- $\Phi_\varepsilon(s) \downarrow s^- := \max\{-s, 0\}$ as $\varepsilon \to 0$ for each $s \in \mathbb{R}$;
- $\Phi_\varepsilon(0) = 0$ and $|\Phi_\varepsilon'(s)| \leq 1$ for all s;
- $\Phi_\varepsilon(s) = s^-$ whenever $|s| \geq \varepsilon$;
- there exists a constant C independent of ε such that

$$\big|s\, \Phi_\varepsilon'(s) - \Phi_\varepsilon(s)\big| \;\leq\; C\varepsilon \qquad \text{for all } s \in \mathbb{R}. \tag{B.4}$$

Such a family is obtained by mollifying the function $s \mapsto s^-$ near $s = 0$ on a scale $O(\varepsilon)$.

Set

$$H_\varepsilon(t, x) := \Phi_\varepsilon\big(h(t, x)\big).$$

Since $h \in C^2(\Omega)$ and $\Phi_\varepsilon \in C^2(\mathbb{R})$, a direct chain-rule computation gives

$$(\partial_t + \partial_x^2)H_\varepsilon = \Phi_\varepsilon'(h)\, (\partial_t + \partial_x^2)h + \Phi_\varepsilon''(h)\, |h_x|^2.$$

Using (B.1), we obtain

$$(\partial_t + \partial_x^2)H_\varepsilon = \Phi_\varepsilon'(h)\,\partial_x(2ph) + \Phi_\varepsilon''(h)\,|h_x|^2.$$

We now rewrite the drift term in divergence form. By the product rule,

$$\partial_x(2p\,\Phi_\varepsilon(h)) = 2p\,\Phi_\varepsilon'(h)\,h_x + 2p_x\,\Phi_\varepsilon(h),$$

while

$$\Phi_\varepsilon'(h)\,\partial_x(2ph) = 2p\,\Phi_\varepsilon'(h)\,h_x + 2p_x\,h\,\Phi_\varepsilon'(h).$$

Subtracting the two expressions yields

$$\Phi_\varepsilon'(h)\,\partial_x(2ph) = \partial_x(2p\,\Phi_\varepsilon(h)) + 2p_x\big(h\,\Phi_\varepsilon'(h) - \Phi_\varepsilon(h)\big).$$

Substituting into the equation for $(\partial_t + \partial_x^2)H_\varepsilon$ we arrive at the identity

$$(\partial_t + \partial_x^2)\Phi_\varepsilon(h) - \partial_x(2p\,\Phi_\varepsilon(h)) = 2p_x\big(h\,\Phi_\varepsilon'(h) - \Phi_\varepsilon(h)\big) + \Phi_\varepsilon''(h)\,|h_x|^2. \tag{B.5}$$

Let $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. Pairing (B.5) with φ and integrating over Ω gives

$$\big\langle (\partial_t + \partial_x^2)\Phi_\varepsilon(h) - \partial_x(2p\,\Phi_\varepsilon(h)), \varphi \big\rangle = I_\varepsilon(\varphi) + J_\varepsilon(\varphi), \tag{B.6}$$

where

$$I_\varepsilon(\varphi) := \int_\Omega 2p_x\big(h\,\Phi_\varepsilon'(h) - \Phi_\varepsilon(h)\big)\varphi\,dx\,dt, \qquad J_\varepsilon(\varphi) := \int_\Omega \Phi_\varepsilon''(h)\,|h_x|^2\,\varphi\,dx\,dt.$$

By (B.4) and smoothness of p_x, φ,

$$|I_\varepsilon(\varphi)| \;\leq\; C\varepsilon \int_\varphi |p_x||\varphi|\,dx\,dt \;\xrightarrow[\varepsilon\to 0]{}\; 0.$$

On the other hand, since Φ_ε is convex we have $\Phi_\varepsilon'' \geq 0$, so

$$J_\varepsilon(\varphi) \;\geq\; 0 \qquad \text{for all } \varepsilon > 0.$$

Thus, for each fixed nonnegative φ, the numbers

$$\mu_\varepsilon(\varphi) := \big\langle (\partial_t + \partial_x^2)\Phi_\varepsilon(h) - \partial_x(2p\,\Phi_\varepsilon(h)), \varphi \big\rangle$$

satisfy

$$\liminf_{\varepsilon\to 0} \mu_\varepsilon(\varphi) \;\geq\; 0.$$

Next we pass to the limit on the left-hand side. Since $\Phi_\varepsilon(s) \downarrow s^-$ for each $s \in \mathbb{R}$ and $|\Phi_\varepsilon(s)| \leq |s^-|$, we have

$$\Phi_\varepsilon(h) \to h^- \quad \text{locally in } L^1(\Omega),$$

and similarly $2p\,\Phi_\varepsilon(h) \to 2p\,h^-$. Hence

$$(\partial_t + \partial_x^2)\Phi_\varepsilon(h) - \partial_x(2p\,\Phi_\varepsilon(h)) \;\to\; (\partial_t + \partial_x^2)h^- - \partial_x(2p\,h^-)$$

in the sense of distributions on Ω, and therefore

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(\varphi) = \big\langle (\partial_t + \partial_x^2)h^- - \partial_x(2p\,h^-), \varphi \big\rangle.$$

Combining this with $\liminf_{\varepsilon\to 0} \mu_\varepsilon(\varphi) \geq 0$ we obtain

$$\big\langle (\partial_t + \partial_x^2)h^- - \partial_x(2p\,h^-), \varphi \big\rangle \;\geq\; 0$$

for every nonnegative $\varphi \in C_c^\infty(\Omega)$. By the Riesz representation theorem, this means that the distribution

$$(\partial_t + \partial_x^2)h^- - \partial_x(2p\,h^-)$$

is a nonnegative Radon measure, which we denote by μ. This is precisely (B.2)–(B.3). $\square$

Remark B.2. In our application we only invoke Lemma B.1 on zero-free regions where E_t is analytic in z and smooth in t, so that $h(t,x,y) = \Im(-E_x/E)$ and $p(t,x,y) = -\Re(E_x/E)$ are smooth. The lemma can also be extended to lower regularity classes (e.g. $h, p \in L^2$ with $\partial_x h \in L^2$) by a standard mollification argument in (t,x), but we do not require this generalisation here.

B.6 Weighted Kato–Carleman integrations by parts

In this subsection we justify rigorously all integrations by parts in Stage 3 that involve the negative part h^- of the imaginary component, the singular drift p, and the spacetime weight

$$W(t,z) = \eta(t)\,\chi(y)\,y^\alpha\,\Theta_{\rho,\varepsilon,\gamma}(t,z)\,\omega_R(x),$$

as in (6.3). The only nontrivial point is integrability near the moving zero set, where $g = -E_x/E$ has $1/r$–type singularities and W vanishes algebraically like r^γ.

We work on a fixed collision-free window $I = [t_1 - \Delta, t_1]$ inside a larger open interval $I^* = (t_1 - \Delta - \tau, t_1 + \tau)$, with $\tau \in (0, \Delta/4)$ as in §6. Fix $Y_0 > 0$ and $R > 0$, and consider the box

$$\Omega = \{(t,x,y) : t \in I^*,\ |x| \le 2R,\ 0 < y \le Y_0\}.$$

All constants below may depend on $I^*, Y_0, R, \alpha, \gamma, \theta$ and the Cartwright data, but are independent of λ, ρ, Δ.

For each fixed $y \in (0, Y_0]$ we regard x and t as the variables and suppress y from the notation. We write

$$h^-(t,x) := \max\{-h(t,x,y), 0\}, \qquad p(t,x) := -\Re\frac{E_x(t,x+iy)}{E_t(x+iy)},$$

and

$$W(t,x) := \eta(t)\,\Theta_{\rho,\varepsilon,\gamma}(t,x+iy)\,\omega_R(x), \qquad w(t,x) := W(t,x)\,h^-(t,x).$$

The lower y–cutoff $\chi(y)$ and the factor y^α play no role in the one-dimensional integrations by parts for each fixed y.

Lemma B.3 (Weighted Kato–Carleman integrations by parts)**.** *Let I^* and Ω be as above, and fix $\gamma > 2$, $\alpha > 2$ and $\varepsilon = \theta\rho$ with $\theta \in (0,1)$. On each collision-free window $I = [t_1 - \Delta, t_1] \subset I^*$ the following hold for each fixed $y \in (0, Y_0]$.*

1. *The functions w, W satisfy*

$$w,\ \partial_x w\ \in L^2(I \times \mathbb{R}_x)$$

and all Carleman integrals of the form

$$\iint_{I\,\mathbb{R}} (t_1 - t)(|\partial_x w|^2 + \phi'(t) w^2) e^{2\phi(t)}\,dx\,dt, \qquad \phi(t) = \frac{\lambda}{t_1 - t},$$

are finite for every $\lambda > 0$.

2. *The distributional Kato inequality from Lemma B.1,*

$$(\partial_t + \partial_x^2)h^- - \partial_x(2p\,h^-) = \mu, \qquad \mu \ge 0,$$

may be tested against

$$\Phi(t,x) := (t_1 - t)\,W(t,x)^2\,e^{2\phi(t)},$$

and all resulting pairings are given by absolutely convergent integrals. In particular, the distributional identity

$$\langle(\partial_t+\partial_x^2)h^- - \partial_x(2p\,h^-), \Phi\rangle = -\iint_{I\times\mathbb{R}} (t_1-t)\,2p\,h^-\,\partial_x(W^2 h^- e^{2\phi})\,dx\,dt + \iint_{I\,\mathbb{R}} (t_1-t)\,\Phi_{xx}\,(h^-)^2\,dx\,dt \tag{B.7}$$

is well-defined, and the right-hand side is finite.

3. *As a consequence, all integrations by parts in x used in the decomposition of J_{drift} and J_{comm} in §6 are legitimate: the boundary terms in x vanish (thanks to ω_R) and all bulk integrals converge absolutely.*

Proof. We prove (i) and (ii) for each fixed $y \in (0, Y_0]$ and then integrate over y; all estimates are uniform in y because y ranges in a compact interval.

Step 1: Local singular behaviour and tube geometry. Let $r(t,x,y)$ denote the (smoothed) distance from $z = x + iy$ to the zero set $Z_I(t)$, and recall from Lemma 5.4 that on the tube $\{r \le 2\rho\}$ one has

$$|h(t,x,y)| \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad |h_x(t,x,y)| \lesssim \frac{1}{r^2} + \frac{1}{\rho^2},$$

and similarly for p, p_x. Outside the tube, Lemma 5.1 gives uniform slab bounds

$$|h| + |p| \lesssim 1 + \rho^{-1}, \qquad |h_x| + |p_x| \lesssim 1 + \rho^{-2}.$$

The tube weight $\Theta_{\rho,\varepsilon,\gamma}$ satisfies the relative-derivative bounds of Lemma 5.3:

$$\frac{|\partial_x \Theta|}{\Theta} \lesssim \frac{1}{r} + \frac{1}{\rho}, \qquad \frac{|\partial_x^2 \Theta|}{\Theta} \lesssim \frac{1}{r^2} + \frac{1}{\rho^2},$$

and the tube integral bookkeeping of Lemma 5.5 gives

$$\int_{r \le 2\rho} w^2 \lesssim 1, \qquad \int_{r \le 2\rho} \frac{w^2}{r^2} \lesssim \rho^{-2}, \qquad \int_{r \le 2\rho} \frac{w^2}{r^4} \lesssim \rho^{-4},$$

provided $\gamma > 2$.

Step 2: L^2–regularity of w and $\partial_x w$. By definition $w = W h^-$ and W is bounded. On the tube,

$$|w| \le |W||h^-| \lesssim \Theta\Big(\frac{1}{r} + \frac{1}{\rho}\Big).$$

Since $\Theta \sim (r/\rho)^\gamma$ for $r \le \rho$ and $\Theta \sim 1$ for $\rho \le r \le 2\rho$, the integrability of w^2 and w^2/r^2 near $r = 0$ follows from Lemma 5.5. Outside the tube, h^- and W are bounded on Ω, so $w \in L^2(I \times \mathbb{R})$.

For $\partial_x w$ we write

$$\partial_x w = W_x h^- + W (h^-)_x.$$

Using the bounds on W_x/W from Lemma 5.3 and on h^-, h^-_x from Lemma 5.4, we obtain on $\{r \le 2\rho\}$:

$$|W_x h^-| \lesssim \Big(\frac{1}{r} + \frac{1}{\rho}\Big) W \Big(\frac{1}{r} + \frac{1}{\rho}\Big), \qquad |W(h^-)_x| \lesssim W\Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big).$$

Squaring and integrating, the tube bounds from Lemma 5.5 (with $\gamma > 2$) show that each contribution is finite. Outside the tube, W_x is bounded and h^-, h^-_x are slab-bounded, so the integrals of $|W_x h^-|^2$ and $|W(h^-)_x|^2$ are finite as well. Thus $w, \partial_x w \in L^2(I \times \mathbb{R})$, and all Carleman bulk integrals in (i) are finite because $(t_1 - t)e^{2\phi}$ and $\phi'(t) e^{2\phi}$ are smooth and bounded on I for each fixed $\lambda > 0$.

Step 3: Admissibility of the weight as a test function. By construction, $\eta \in C_c^\infty(I^*)$, $\omega_R \in C_c^\infty(\mathbb{R})$, and $\Theta_{\rho,\varepsilon,\gamma}$ is smooth in (t,x) on Ω. Thus

$$\Phi(t,x) = (t_1 - t)\, W(t,x)^2\, e^{2\phi(t)}$$

is a C^∞ function compactly supported in $I^* \times \mathbb{R}_x$ and in particular in Ω for each fixed y. Hence Φ is a legitimate test function for the distribution

$$(\partial_t + \partial_x^2) h^- - \partial_x(2p\, h^-) = \mu \ge 0$$

appearing in Lemma B.1.

By the definition of distributional derivatives,

$$\langle(\partial_t+\partial_x^2)h^-,\Phi\rangle = -\iint h^-\,(\partial_t\Phi+\partial_x^2\Phi)\,dx\,dt,$$
$$\langle\partial_x(2p\,h^-),\Phi\rangle = -\iint 2p\,h^-\,\partial_x\Phi\,dx\,dt,$$

provided the integrals on the right-hand side are absolutely convergent. We now check this.

On $\{r\le 2\rho\}$ the derivatives of W satisfy

$$\frac{|W_x|}{W}+\frac{|W_{xx}|}{W}+\frac{|W_t|}{W}\ \lesssim\ \frac{1}{r}+\frac{1}{\rho},$$

by Lemma 5.3 and the structure of η,ω_R. Hence $\partial_x\Phi$, $\partial_x^2\Phi$ and $\partial_t\Phi$ can be bounded pointwise by

$$|\partial_x\Phi|+|\partial_x^2\Phi|+|\partial_t\Phi|\ \lesssim\ (t_1-t)\,e^{2\phi}\,W^2\Big(\frac{1}{r^2}+\frac{1}{\rho^2}\Big),$$

up to harmless factors depending on λ and I^*. Together with the bounds on h^-,p from Step 1, we obtain on $\{r\le 2\rho\}$:

$$|h^-\,(\partial_t\Phi+\partial_x^2\Phi)|\ \lesssim\ W^2\Big(\frac{1}{r}+\frac{1}{\rho}\Big)\Big(\frac{1}{r^2}+\frac{1}{\rho^2}\Big),$$

and

$$|2p\,h^-\,\partial_x\Phi|\ \lesssim\ W^2\Big(\frac{1}{r}+\frac{1}{\rho}\Big)^2\Big(\frac{1}{r}+\frac{1}{\rho}\Big).$$

Expanding and using Lemma 5.5 shows that each term is integrable on the tube: every factor r^{-k} is compensated by W^2, whose algebraic vanishing produces powers $r^{2\gamma}$ with $\gamma>2$.

Outside the tube, W,W_x,W_{xx},W_t are bounded and so are h^-,p, hence $h^-\,(\partial_t\Phi+\partial_x^2\Phi)$ and $p\,h^-\,\partial_x\Phi$ are bounded and compactly supported in (t,x), and thus integrable. Therefore both pairings

$$\langle(\partial_t+\partial_x^2)h^-,\Phi\rangle,\qquad \langle\partial_x(2p\,h^-),\Phi\rangle$$

are given by absolutely convergent integrals, and the distributional Kato inequality applied to Φ yields

$$\langle(\partial_t+\partial_x^2)h^- - \partial_x(2p\,h^-),\Phi\rangle\ge 0.$$

Unwinding the definitions, this is exactly (B.7), which proves (ii).

Step 4: Boundary terms in x. In all the weighted integrations by parts in x in Stage 3, the boundary terms at spatial infinity vanish because ω_R has compact support and W therefore vanishes for $|x|$ large. There are no boundary terms at the moving zeros: these are handled by the distributional formulation and the integrability just established. Thus every integration by parts in x used to derive the expressions J_{drift} and J_{comm} in §6 is legitimate, which proves (iii). □

C Local normal form near a collision and uniform p, p_x bounds

In this subsection we record a local description of E_t near a collision time and use it to upgrade the singular profile bounds for p and p_x so that they remain uniform up to a collision time t^*.

Lemma C.1 (Local normal form near a collision)**.** *Let $t^* \in (0, \infty)$ and suppose that $z^* \in \mathbb{C}$ is a zero of E_{t^*} of multiplicity $m \geq 2$. Then there exist:*

- *radii $0 < r_1 < r_0$,*
- *a time radius $\delta > 0$,*
- *holomorphic functions $z_1(t), \ldots, z_m(t)$ on $(t^* - \delta, t^* + \delta)$,*
- *and a holomorphic function $G(t, z)$ on $(t^* - \delta, t^* + \delta) \times B(z^*, r_0)$,*

such that:

1. *For each $t \in (t^* - \delta, t^* + \delta)$, the only zeros of E_t in $B(z^*, r_0)$ are $z_1(t), \ldots, z_m(t)$ (counted with multiplicity), and*
$$E_t(z) = \prod_{j=1}^{m} (z - z_j(t))\, G(t, z),$$
with $G(t, z) \neq 0$ on $(t^* - \delta, t^* + \delta) \times B(z^*, r_0)$.
2. *At $t = t^*$ one has $z_j(t^*) = z^*$ for all $j = 1, \ldots, m$.*

Moreover, the functions $z_j(t)$ and $G(t, z)$ depend real-analytically on t.

Proof. Consider $F(t, z) := E_t(z)$, which is entire in z and real-analytic in t, and in fact holomorphic in (t, z) after complexifying t. Since z^* is a zero of multiplicity m of $F(t^*, \cdot)$, the classical Weierstrass preparation theorem (applied in (t, z) near (t^*, z^*)) provides a neighbourhood

$$U = (t^* - \delta, t^* + \delta) \times B(z^*, r_0)$$

and holomorphic functions $a_0(t), \ldots, a_{m-1}(t)$, $H(t, z)$, with $H(t, z) \neq 0$ on U, such that

$$F(t, z) = P_t(z)\, H(t, z),$$

where
$$P_t(z) = z^m + a_{m-1}(t) z^{m-1} + \cdots + a_0(t)$$
is a monic degree-m polynomial in z whose roots are precisely the zeros of $F(t, \cdot)$ in $B(z^*, r_0)$, counted with multiplicity.

By factoring P_t into linear factors,

$$P_t(z) = \prod_{j=1}^{m} (z - z_j(t)),$$

we obtain holomorphic root functions $z_1(t), \ldots, z_m(t)$ (after analytic continuation in t and relabelling the branches if necessary). At $t = t^*$ all m roots coalesce at z^*, since z^* is a zero of multiplicity m of $F(t^*, \cdot)$.

Setting $G(t,z) := H(t,z)$ gives the desired factorisation

$$E_t(z) = F(t,z) = \prod_{j=1}^{m} (z - z_j(t))\, G(t,z)$$

on U, with $G(t,z) \neq 0$ there. Holomorphic dependence of $a_k(t)$ and $H(t,z)$ implies real-analytic dependence of $z_j(t)$ and $G(t,z)$ on t. $\square$

We now use this normal form to obtain p, p_x bounds that are uniform up to t^*.

Lemma C.2 (Uniform local singular profile up to a collision)**.** *Let t^* and z^* be as in Lemma C.1, and let U be the corresponding neighbourhood with factorisation*

$$E_t(z) = \prod_{j=1}^{m} (z - z_j(t))\, G(t,z).$$

Fix $0 < r \le r_1$ where r_1 is as in Lemma C.1, and define the local distance

$$r(t,z) := \big(z, \{z_1(t), \dots, z_m(t)\}\big).$$

Then there exists a constant $C < \infty$, depending only on m, r_0, r_1, δ and on $\sup_U(|G| + |\partial_z G| + |\partial_z^2 G|)$, such that for all $(t,z) \in U$ with $0 < r(t,z) \le r$ one has

$$\Big|-\Re\frac{E_t'(z)}{E_t(z)}\Big| \;\le\; C\Big(\frac{1}{r(t,z)} + 1\Big), \qquad \Big|\partial_x\Big(-\Re\frac{E_t'(z)}{E_t(z)}\Big)\Big| \;\le\; C\Big(\frac{1}{r(t,z)^2} + 1\Big). \tag{C.1}$$

In particular, if $\rho \le r_1$ and we restrict to the tube $\{(t,z) \in U : r(t,z) \le 2\rho\}$, then

$$|p(t,z)| \;\lesssim\; \frac{1}{r(t,z)} + \frac{1}{\rho}, \qquad |p_x(t,z)| \;\lesssim\; \frac{1}{r(t,z)^2} + \frac{1}{\rho^2},$$

with implicit constants independent of t up to and including $t = t^*$.

Proof. Differentiate the factorisation

$$E_t(z) = P_t(z)\, G(t,z), \qquad P_t(z) := \prod_{j=1}^{m} (z - z_j(t)).$$

Then

$$\frac{E_t'(z)}{E_t(z)} = \frac{P_t'(z)}{P_t(z)} + \frac{G_z(t,z)}{G(t,z)}.$$

The logarithmic derivative of P_t is explicitly

$$\frac{P_t'(z)}{P_t(z)} = \sum_{j=1}^{m} \frac{1}{z - z_j(t)}.$$

Let $r = r(t,z) = \min_j |z - z_j(t)|$. Then for each j, $|z - z_j(t)| \ge r$, so

$$\Big|\frac{P_t'(z)}{P_t(z)}\Big| \le \sum_{j=1}^{m} \frac{1}{|z - z_j(t)|} \le \frac{m}{r}.$$

Thus

$$\Big|-\Re\frac{P_t'(z)}{P_t(z)}\Big| \;\le\; \frac{m}{r}.$$

Since $G(t,z) \neq 0$ on U and U is relatively compact, there is a uniform bound

$$\sup_{(t,z)\in U} \Big|\frac{G_z(t,z)}{G(t,z)}\Big| \;\le\; C_G < \infty.$$

Combining these estimates gives

$$\Big|-\Re\frac{E_t'(z)}{E_t(z)}\Big| \le \frac{m}{r} + C_G \le C\Big(\frac{1}{r}+1\Big),$$

for some constant C depending only on m and C_G. This is the first inequality in (C.1).

For the derivative in x, note that ∂_x coincides with ∂_z on holomorphic functions, so

$$\partial_x\Big(\frac{P_t'(z)}{P_t(z)}\Big) = \partial_z\Big(\sum_{j=1}^{m}\frac{1}{z-z_j(t)}\Big) = -\sum_{j=1}^{m}\frac{1}{(z-z_j(t))^2},$$

and therefore

$$\Big|\partial_x\frac{P_t'(z)}{P_t(z)}\Big| \le \sum_{j=1}^{m}\frac{1}{|z-z_j(t)|^2} \le \frac{m}{r^2}.$$

Similarly, on U we have a uniform bound for the holomorphic function G_z/G and its z–derivative:

$$\sup_{(t,z)\in U}\Big|\partial_x\Big(\frac{G_z}{G}\Big)\Big| = \sup_{(t,z)\in U}\Big|\partial_z\Big(\frac{G_z}{G}\Big)\Big| \le C_G'.$$

Thus

$$\Big|\partial_x\Big(-\Re\frac{E_t'(z)}{E_t(z)}\Big)\Big| \le \frac{m}{r^2} + C_G' \le C\Big(\frac{1}{r^2}+1\Big),$$

which is the second inequality in (C.1).

Finally, if we restrict to the tube region $r \le 2\rho$ with $\rho \le r_1$, then $1 \lesssim \rho^{-1}$ and $1 \lesssim \rho^{-2}$, so the $+1$ terms can be absorbed into ρ^{-1} and ρ^{-2} respectively, giving

$$|p(t,z)| \;\lesssim\; \frac{1}{r}+\frac{1}{\rho}, \qquad |p_x(t,z)| \;\lesssim\; \frac{1}{r^2}+\frac{1}{\rho^2},$$

with constants independent of t up to and including $t = t^*$. □

Remark C.3. The key point of Lemma C.2 is that the singular profile of p and p_x near the zero set is controlled purely by the number m of colliding zeros and the local scale $r(t,z)$, and does *not* blow up as $t \to t^*$. In particular, the same $1/r$ and $1/r^2$ behaviour persists uniformly up to a collision time. Outside a small disc around the collision, the original Cartwright-based Lemma 5.4 applies unchanged, so together these give a global local profile for p, p_x that is uniform on compact time intervals containing t^*.

D Constants and parameter dependence

Throughout the paper we write $C > 0$ for a harmless constant whose value may change from line to line. When the dependence on parameters matters we write, for example, $C(J, Y_0, \gamma)$ to indicate dependence on the time interval J, the height parameter Y_0, and the tube exponent γ.

Unless explicitly stated otherwise, all constants depend only on the following *global data*:

- the fixed Cartwright entire function Ξ and its Cartwright data (A_Ξ, B_Ξ) as in (4.2);
- a fixed compact time interval $J \Subset (0, \infty)$ on which we work;
- a fixed vertical height $Y_0 > 0$;
- the tube exponent $\gamma > 2$ and the weight exponent $\alpha > 2$;
- the fixed smoothing kernel $\eta \in C_c^\infty(B(0,1))$ used to define δ_ε;
- the fixed cutoff profiles in t, x, y (the time cutoff $\eta(t)$, the vertical cutoff $\chi(y)$, and the horizontal cutoff $\omega_R(x)$).

In particular, once these are fixed, the constants appearing in the lemmas below do *not* depend on the local geometric parameters

$$R \geq 1, \qquad 0 < \rho \leq 1, \qquad \varepsilon = \theta\rho, \quad 0 < \theta < 1,$$

nor on the Carleman parameters λ, Δ, except where this dependence is explicitly recorded (for instance in the cubic coupling $\rho = \lambda^{-1/2}$, $\Delta \lesssim \min\{\lambda^{-1/3}, L_I^{-2/3}\}$).

For the reader's convenience we summarize the dependence of the main constants, arranged by stage and referencing the lemmas used in the proof of Theorem 2.1.

Cartwright bounds and zero motion (Stage 1).

- *Uniform Cartwright growth (U1), Lemma 4.3.* For each compact $J \Subset (0, \infty)$ there exists $C_J = C_J(J, \Xi) > 0$ such that

  $$|E_t(z)| \leq C_J \exp\bigl(A_\Xi(1 + |z|)\log(2 + |z|)\bigr) \qquad (t \in J,\ z \in \mathbb{C}).$$

  The constant C_J depends only on J and the Cartwright data of Ξ, not on Y_0, R, ρ.

- *Rectangle zero counting (U2), Lemma 4.4.* For each compact $J \Subset (0, \infty)$ there exist constants $K_J, \widetilde{C}_J > 0$, depending only on J and Ξ, such that for all $t \in J$ and all $R, H \geq 1$,

  $$N_t(R; H) \leq K_J + \widetilde{C}_J(R + H)\log(2 + R + H),$$

  where $N_t(R; H)$ counts the zeros of E_t in $\{|\Re z| \leq R, |\Im z| \leq H\}$.

- *Collision discreteness, Lemma 4.5.* For each compact interval $J \Subset (0, \infty)$ and each bounded spatial region $\Omega \subset \mathbb{C}$ there is an integer $N_{\mathrm{coll}}(J, \Omega) < \infty$, depending only on J, Ω and the Cartwright data of Ξ, such that at most $N_{\mathrm{coll}}(J, \Omega)$ collision times occur in J with zeros in Ω.

- *Local speed bound, Lemma 4.7.* On a collision-free window $I = [\underline{t}, \overline{t}] \subset J$ and in a box $\Omega_{Y_0,2R}$ there exists a speed bound $L_I = L_I(I, Y_0, R, \Xi)$ such that

  $$\sup_{t \in I} \sup_{\rho_k(t) \in \Omega_{Y_0,2R}} |\dot{\rho}_k(t)| \leq L_I.$$

  This constant depends only on the global data and on I, Y_0, R, not on ρ, λ, Δ.

Tubes, smoothed distance and local profiles (Stage 2).

- *Slab bounds away from the zero set, Lemma 5.1.* For each compact $I \Subset (0, \infty)$ and fixed Y_0, R there exists $C_{\text{slab}} = C_{\text{slab}}(I, Y_0, R)$ such that for all $\rho \in (0, 1]$, all $t \in I$ and all $z \in \Omega_{Y_0,2R}$ with $(z, Z_t) \geq \rho$,

$$|p(t, z)| \leq C_{\text{slab}}(1 + \rho^{-1}), \qquad |p_x(t, z)| \leq C_{\text{slab}}(1 + \rho^{-2}),$$

  and similarly for h, h_x. The constant C_{slab} is independent of ρ.

- *Smoothed distance bounds, Lemma 5.2.* On a collision-free window I in $\Omega_{Y_0,2R}$ there is a constant $C_I = C_I(I, Y_0, R, \eta)$ such that

$$\|\nabla \delta_\varepsilon\|_\infty \leq 1, \quad \|\nabla^2 \delta_\varepsilon\|_\infty \leq C_I\, \varepsilon^{-1}, \quad |\partial_t \delta_\varepsilon| \leq L_I.$$

  Here C_I depends on I, Y_0, R and the fixed mollifier η, but not on ρ beyond the explicit factor ε^{-1}.

- *Relative-derivative bounds for the tube, Lemma 5.3.* For the tube weight $\Theta_{\rho,\varepsilon,\gamma}$ with fixed $\gamma > 2$ and $\varepsilon = \theta\rho$ $(0 < \theta < 1)$, there is a constant $C_\Theta = C_\Theta(\gamma, \theta, C_I)$ such that on $\Omega_{Y_0,2R} \times I$,

$$\frac{|\partial_x \Theta|}{\Theta} \leq C_\Theta\Big(\frac{1}{r} + \frac{1}{\rho}\Big), \qquad \frac{|\partial_x^2 \Theta|}{\Theta} \leq C_\Theta\Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big), \qquad \frac{|\partial_t \Theta|}{\Theta} \leq C_\Theta\Big(\frac{L_I}{r}\Big).$$

- *Local singular profile on the tube, Lemma 5.4.* On $\{r \leq 2\rho\}$ there is a constant $C_{\text{tube}} = C_{\text{tube}}(J, Y_0, R, \Xi)$ such that

$$|g(t, z)| \leq C_{\text{tube}}\Big(\frac{1}{r} + \frac{1}{\rho}\Big), \qquad |g_x(t, z)| \leq C_{\text{tube}}\Big(\frac{1}{r^2} + \frac{1}{\rho^2}\Big),$$

  and hence the same bounds for p, h, p_x, h_x. This constant is uniform in $t \in J$ and does not depend on ρ, λ, Δ.

- *Tube integral bookkeeping, Lemma 5.5.* For the weight W used in the barrier (with fixed $\alpha > 2$ and $\gamma > 2$), there is a constant $C_{\text{tube-int}} = C_{\text{tube-int}}(Y_0, \alpha, \gamma)$ such that on each tube $\{r \leq 2\rho\}$,

$$\int_{r \leq 2\rho} w^2 \ \leq C_{\text{tube-int}}, \qquad \int_{r \leq 2\rho} \frac{w^2}{r^2} \ \leq C_{\text{tube-int}}\rho^{-2}, \qquad \int_{r \leq 2\rho} \frac{w^2}{r^4} \ \leq C_{\text{tube-int}}\rho^{-4}.$$

  The constant $C_{\text{tube-int}}$ is independent of ρ, λ, Δ.

Carleman–Kato barrier (Stage 3).

- *Backward Carleman identity, Lemma 6.1.* This is an exact identity with no implicit constants; it holds for all smooth, compactly supported w supported in $[t_1 - \Delta, t_1]$.

- *Weak Kato inequality, Lemma B.1.* The measure μ in

$$(\partial_t + \partial_x^2)h^- - \partial_x(2p\, h^-) = \mu, \qquad \mu \geq 0,$$

  is determined canonically by h, p and involves no additional constants.

- *Weighted Kato–Carleman integrations by parts, Lemma B.3.* The justification of all x–integrations by parts for $w = Wh^-$ uses only the tube bounds above and produces implicit constants depending on $J, Y_0, \alpha, \gamma, \theta$ (and the Cartwright data of Ξ) but not on R, ρ, λ, Δ.

- *Quantitative Carleman absorption, Lemma 6.4.* There exists a constant $C_0 = C_0(J, Y_0, \alpha, \gamma, \theta)$ such that for all admissible parameters one has

$$B(w) \;\le\; C_0 \int_{t_1-\Delta}^{t_1}\int_{\mathbb{R}} (t_1 - t)\, A(\lambda, \rho, \Delta, L_I)\, w^2 e^{2\phi}\, dx\, dt,$$

  with A as in (6.19). Under the cubic coupling $\rho = \lambda^{-1/2}$ and $\Delta \le c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}\}$, with c_0 and a threshold λ_* depending only on $(J, Y_0, \alpha, \gamma, \theta)$, this implies $B(w) = 0$.

- *Short-time barrier, Proposition 6.3.* The constants λ_0 and c_0 appearing in the statement of the short-time barrier can be chosen depending only on the global data $(J, Y_0, \alpha, \gamma, \theta)$ and on the fixed cutoffs. They are uniform over all collision-free windows $I \subset J$ and all boxes $\Omega_{Y_0,2R}$, i.e. independent of R, ρ and of the particular configuration of zeros inside $\Omega_{Y_0,2R}$ (except through the local speed bound L_I entering the coupling).

Collision bridging (Stage 4).

- *Tube energy and absolute continuity, Lemma 7.1.* On a collision-free interval I_0 the tube energy

$$\mathcal{E}(t) = \iint ((h^-)^2 + |(h^-)_x|^2) W^2\, dx\, dy$$

  satisfies a differential inequality

$$|\mathcal{E}'(t)| \le C_{\mathrm{AC}}(I_0, Y_0, R, \alpha, \gamma, \theta)\, (1 + L_{I_0}^2 + \rho^{-4})\, \mathcal{E}(t)$$

  for a.e. $t \in I_0$. The constant C_{AC} is independent of λ, Δ.

- *Local normal form and uniform profile at a collision, Lemma C.1 and Lemma C.2.* The constants in the holomorphic factorisation

$$E_t(z) = \prod_{j=1}^{m} (z - z_j(t))\, G(t, z)$$

  and in the uniform bounds

$$|p(t, z)| \lesssim r^{-1} + 1, \qquad |p_x(t, z)| \lesssim r^{-2} + 1$$

  near a collision depend only on m and on the sup norms of G and its first two z–derivatives on a small neighbourhood of the collision. Ultimately these sup norms can be bounded in terms of the Cartwright data of Ξ and J, and are independent of ρ, λ, Δ.

- *Collision bridge, Lemma 7.3.* No new constants are introduced beyond those controlling $\mathcal{E}$ in Lemma 7.1. The conclusion that positivity propagates across t^* is therefore uniform in ρ, R, λ, Δ once the global data are fixed.

Exhaustion and global positivity (Stage 5).

- *Uniformity of local bounds in R and ρ, Lemma 8.2.* The constants C_{slab}, C_{tube}, C_Θ, C_W appearing in the slab, tube and weight derivative bounds can be chosen depending only on $(J, Y_0, \alpha, \gamma, \theta)$ and the Cartwright data, and are uniform in $R \ge 1$ and $0 < \rho \le 1$.

- *Uniform Carleman absorption in the tube regime, Lemma 8.3.* There exist $\lambda_0, c_0 > 0$ depending only on the global data such that the short-time barrier of Proposition 6.3 holds for all windows $I \subset J$, all boxes $\Omega_{Y_0,2R}$ and all $0 < \rho \le 1$ satisfying the cubic coupling $\rho = \lambda^{-1/2}$, $\Delta \le c_0 \min\{\lambda^{-1/3}, L_I^{-2/3}\}$. In particular, the barrier constants are uniform in R and ρ.

- *Exhaustion in x and y, Lemma 8.1.* This lemma is geometric and uses no new analytic constants: once $h^-(t,z) = 0$ wherever $W_n(t,z) \neq 0$ for a sequence of weights with $R_n \to \infty$ and $\rho_n \downarrow 0$, we conclude that $h^-(t,z) \equiv 0$ on all of $\mathbb{C}_+$.

Pick positivity and endgame (Stage 6).

- *Nevanlinna no-poles lemma, Lemma 9.1.* This is a purely local statement about meromorphic functions on $\mathbb{C}_+$ with nonnegative imaginary part and involves no parameters.

In summary, after fixing the global data $(\Xi, J, Y_0, \alpha, \gamma, \theta)$ and the cutoff profiles, all constants appearing in the estimates of Stages 1–6 can be chosen uniformly in the geometric parameters (R, ρ) and in the Carleman parameters (λ, Δ), subject only to the explicit cubic coupling conditions recorded in Lemmas 8.3 and 6.4.

E Companion diagrams for the Xi flow

The figures in this section play no role in the proofs. They are included solely as illustrative diagrams for the heat deformation

$$E_t(z) = e^{-t\partial_z^2}\Xi(z),$$

which we refer to in this paper as the *Xi flow*, and for the weights appearing in the backward positivity argument for the log-derivative $g(t,z) = -E_t'(z)/E_t(z)$.

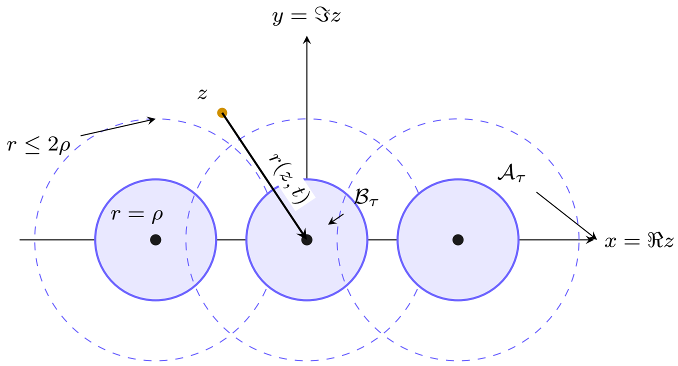

Figure 2: **Stage 2 tube geometry (Lemmas 5.2–5.5).**

Figure 2 shows a horizontal slice at a fixed time t. The black dots are schematic real zeros of E_t on the real axis. Around each zero we draw a disc of radius ρ and a dashed disc of radius 2ρ, forming a tube around the moving zero set $Z_I(t)$. The arrow from z to the nearest zero represents the smoothed distance $r(z,t)$. The region $\mathcal{B}_\tau = \{\Theta_{\rho,\varepsilon,\gamma} < \tau\}$ is drawn inside the inner discs, while $\mathcal{A}_\tau$ corresponds to the plateau outside all dashed circles where $\Theta_{\rho,\varepsilon,\gamma} \approx 1$ and the Stage 1 bounds control p and p_x. The precise definitions and estimates appear in Lemmas 5.2–5.5; the figure is only a geometric aid to visualising Stage 2.

Stages 3–5 companion picture (Carleman–Kato barrier)

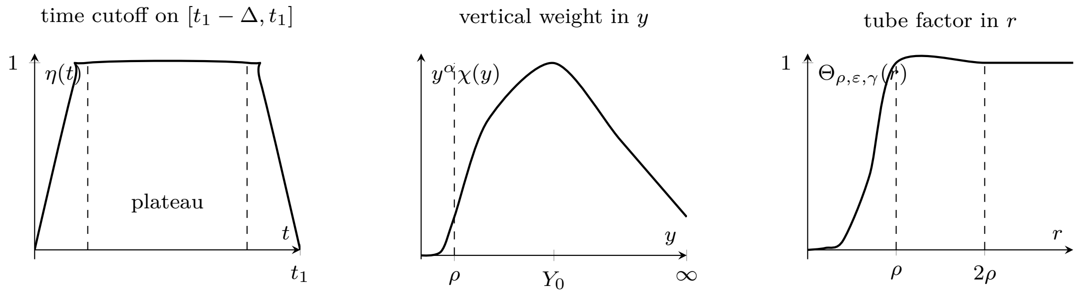

Figure 3: **Weight profiles for Stages 3–5 (Lemmas 6.1–8.1).**

Figure 3 shows the one-dimensional building blocks of the weight

$$W(t,z) = \eta(t)\,\chi(y)\,y^\alpha\,\Theta_{\rho,\varepsilon,\gamma}(t,z)\,\omega_R(x)$$

used in the backward Carleman–Kato argument. Panel (a) depicts the time cutoff $\eta(t)$ on the backward window $[t_1 - \Delta, t_1]$, with thin shoulders and a flat interior plateau (Stage 3). Panel (b) shows the vertical factor $y^\alpha\chi(y)$ for $\alpha > 2$, localising mass to an intermediate strip in y and supporting the exhaustion in Stage 5. Panel (c) shows the tube factor $\Theta_{\rho,\varepsilon,\gamma}(r)$ from Stages 2–3, which vanishes inside the tube $r \le \rho$ and is essentially 1 for $r \ge 2\rho$. Together these schematic profiles summarise the role of W in Lemmas 6.1–8.1; the figure is included only as a visual guide to Stages 3–5.

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